

On the Structure and Representations of Max-Stable Processes*

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Abstract

We develop classification results for max-stable processes, based on their spectral representations. The structure of max-linear isometries and minimal spectral representations play important roles. We propose a general classification strategy for measurable max-stable processes based on the notion of co-spectral functions. In particular, we discuss the spectrally continuous-discrete, the conservative-dissipative, and positive-null decompositions. For stationary max-stable processes, the latter two decompositions arise from connections to non-singular flows and are closely related to the classification of stationary sum-stable processes. The interplay between the introduced decompositions of max-stable processes is further explored. As an example, the Brown-Resnick stationary processes, driven by fractional Brownian motions, are shown to be dissipative. A result on general Gaussian processes with stationary increments and continuous paths is obtained.

1 Introduction

Max-stable processes have been studied extensively in the past 30 years. The works of Balkema and Resnick [2], de Haan [6, 7], de Haan and Pickands [8], Giné *et al.* [10] and Resnick and Roy [25], among many others have lead to a wealth of knowledge on max-stable processes. The seminal works of de Haan [7] and de Haan and Pickands [8] laid the foundations of the spectral representations of max-stable processes and established important structural results for stationary max-stable processes. Since then, however, while many authors focused on various important aspects of max-stable processes, the general theory of their representation and structural properties had not been thoroughly explored. At the same time, the structure and the classification of sum-stable processes has been vigorously studied. Rosiński [27], building on the seminal works of Hardin [12, 13] about minimal representations, developed the important connection between stationary sum-stable processes and flows. This lead to a number of important contributions on the structure of sum-stable processes (see, e.g. [30, 28, 22, 23, 31]). There are relatively few results of this nature about the structure of max-stable processes, with the notable exceptions of de Haan and Pickands [8], Davis and Resnick [5] and the very recent works of Kabluchko *et al.* [16] and Kabluchko [15].

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Our goal here is to develop representation and classification theory for max-stable processes, similar to the available one for sum-stable processes. We are motivated by the strong similarities between the spectral representations of sum- and max-stable processes. This procedure however, is non-trivial. The notion of *minimal extremal integral* representation plays a key role as does the *minimal integral representation* for α -stable processes (see Hardin [13] and Rosiński [27, 29]). Before one can fruitfully handle the *minimal extremal integral* representations, it turns out that one should first thoroughly investigate the structure of max-linear isometries, also known as the *pistons* of de Haan and Pickands [8]. We refine and extend their work in Section 3. In Section 4, we develop the theory of minimal representations for max-stable processes. Our approach is motivated by the works of Hardin [13] and Rosiński [27] in the sum-stable context.

In Section 5, we establish general classification results for max-stable processes by using the developed theory of *minimal spectral representations*. In Section 5.1, we first show that essentially any max-stable process can be represented uniquely as the maximum of two independent components, characterized as *spectrally continuous* and *spectrally discrete*, respectively. The spectrally discrete part gives rise to the notion of *discrete principal components*, which may be of independent interest in modeling of max-stable processes and fields.

In Section 5.2, we introduce the notion of *co-spectral functions*, for the large class of measurable max-stable processes $X = \{X_t\}_{t \in T}$. There T is a separable metric space equipped with the Borel- σ -algebra and a σ -finite measure. The co-spectral functions of such processes are invariant to the choice of the spectral representations, up to a multiplicative factor. This allows us to develop a general strategy for the classification of measurable α -Fréchet processes, based on positive cones of co-spectral functions. As particular examples, we obtain the *conservative-dissipative* and *positive-null* decompositions, which correspond to certain choices of cones for the co-spectral functions.

Section 6 is devoted to the classification of stationary max-stable processes. As in the sum-stable case, the minimal representations allow us to associate a measurable non-singular flow to every measurable stationary max-stable process. This correspondence enables one to apply existing ergodic theory results about the flow to characterize the max-stable process. The conservative-dissipative and positive-null decompositions introduced in Examples 5.3 and 5.4 are in fact motivated by the corresponding decompositions of the underlying flow. These two results are in close correspondence with the classifications of Rosiński [27] and Samorodnitsky [31] for sum-stable processes. As in Rosiński [27], we obtain that the class of stationary max-stable processes generated by dissipative flows is precisely the class of mixed moving maxima.

In Section 7, we apply the results in Section 6 to Brown-Resnick processes. We give simple necessary and sufficient conditions for a generalized Brown-Resnick stationary process to be a mixed moving maxima. This extends and complements the recent results of Kabluchko *et al.* [16]. In fact, as a by-product, by combining our results and those in [16], we obtain an interesting fact about general zero-mean Gaussian processes $W = \{W_t\}_{t \in \mathbb{R}}$ with stationary increments and continuous paths. Namely, for such processes, we have that, with probability one,

$$\lim_{|t| \rightarrow \infty} \left(W_t - \text{Var}(W_t)/2 \right) = -\infty \text{ implies } \int_{\mathbb{R}} \exp\{W_t - \text{Var}(W_t)/2\} dt < \infty.$$

In particular, we show that if $\{W_t\}_{t \in \mathbb{R}}$ is a fractional Brownian motion, then the gen-

erated Brown–Resnick process is a mixed moving maxima. We conclude Section 7 with some open questions. Some proofs and auxiliary results are given in the Appendix.

Part of our results in Sections 5 and 6 are modifications and extensions of results of de Haan and Pickands [8]. The main difference is that we provide a complete treatment of the measurability issue, when the processes are continuously indexed. Before we proceed with the more technical preliminaries, we are obliged to mention the recent work of Kabluchko [15]. In this exciting contribution, the author establishes some very similar classification results by using an *association device* between max– and sum–stable processes. This association allows one to transfer existing classifications of sum–stable processes to the max–stable domain. It also clarifies the connection between these two classes of processes. Our results were obtained independently and by using rather different technical tools. The combination of the two approaches provides a more clear picture on the structure of max– and sum–stable processes as well as their interplay.

2 Preliminaries

The importance of max–stable processes stems from the fact that they arise in the limit of the component–wise maxima of independent processes. It is well known that the univariate marginals of a max–stable process are necessarily extreme value distributions, i.e. up to rescaling and shift they are either Fréchet, Gumbel or negative Fréchet. The dependence structure of the max–stable processes, however, can be quite intricate and it does not hinge on the extreme value type of the marginal distributions (see e.g. Proposition 5.11 in Resnick [24]). Therefore, for convenience and without loss of generality we will focus here on max–stable process with Fréchet marginal distributions. Recall that a positive random variable $Z \geq 0$ has α –Fréchet distribution, $\alpha > 0$, if

$$\mathbb{P}(Z \leq x) = \exp\{-\sigma^\alpha x^{-\alpha}\}, x \in (0, \infty).$$

Here $\|Z\|_\alpha := \sigma > 0$ stands for the *scale coefficient* of Z . It turns out that a stochastic process $\{X_t\}_{t \in T}$ with α –Fréchet marginals is max–stable *if and only if* all positive *max–linear combinations*:

$$\max_{1 \leq j \leq n} a_j X_{t_j} \equiv \bigvee_{1 \leq j \leq n} a_j X_{t_j} \quad \forall a_j > 0, t_j \in T, 1 \leq j \leq n, \quad (2.1)$$

are α –Fréchet random variables (see de Haan [6] and e.g. [35]). This feature resembles the definition of Gaussian or, more generally, symmetric α –stable (sum–stable) processes, where all finite–dimensional linear combinations are univariate Gaussian or symmetric α –stable, respectively (see e.g. [32]). We shall therefore refer to the max–stable processes with α –Fréchet marginals as to *α –Fréchet processes*.

The seminal work of de Haan [7] provides convenient *spectral representations* for stochastically continuous α –Fréchet processes in terms of functionals of Poisson point processes on $(0, 1) \times (0, \infty)$. Here, we adopt the slightly more general, but essentially equivalent, approach of representing max–stable processes through extremal integrals with respect to a random sup–measures (see Stoev and Taqqu [35]). We do so in order to emphasize the analogies with the well–developed theory of sum–stable processes (see e.g. Samorodnitsky and Taqqu [32]).

Definition 2.1. Consider a measure space (S, \mathcal{S}, μ) and suppose $\alpha > 0$. A stochastic process $\{M_\alpha(A)\}_{A \in \mathcal{S}}$, indexed by the measurable sets $A \in \mathcal{S}$ is said to be an *α –Fréchet*

random sup-measure with control measure μ , if the following conditions hold:

- (i) the $M_\alpha(A_i)$'s are independent for disjoint $A_i \in \mathcal{S}$, $1 \leq i \leq n$.
- (ii) $M_\alpha(A)$ is α -Fréchet with scale coefficient $\|M_\alpha(A)\|_\alpha = \mu(A)^{1/\alpha}$.
- (iii) for all disjoint A_i 's, $i \in \mathbb{N}$, we have $M_\alpha(\cup_{i \in \mathbb{N}} A_i) = \bigvee_{i \in \mathbb{N}} M_\alpha(A_i)$, almost surely.

Now, given an α -Fréchet random sup-measure M_α as above, one can define the *extremal integral* of a non-negative simple function $f(u) := \sum_{i=1}^n a_i 1_{A_i}(u) \geq 0$, $A_i \in \mathcal{S}$:

$$\int_S^e f dM_\alpha \equiv \int_S^e f(u) M_\alpha(du) := \bigvee_{1 \leq i \leq n} a_i M_\alpha(A_i).$$

The resulting extremal integral is an α -Fréchet random variable with scale coefficient $(\int_E f^\alpha d\mu)^{1/\alpha}$. The definition of $\int_S^e f dM_\alpha$ can, by continuity in probability, be naturally extended to integrands f in the space

$$L_+^\alpha(S, \mu) := \left\{ f : S \rightarrow \mathbb{R}_+ : f \text{ measurable with } \int_S f^\alpha d\mu < \infty \right\}.$$

It turns out that the random variables $\xi_j := \int_S^e f_j dM_\alpha$, $1 \leq j \leq n$ are independent if and only if the f_j 's have pairwise disjoint supports (mod μ). Furthermore, the extremal integral is *max-linear*:

$$\int_S^e (af \vee bg) dM_\alpha = a \int_S^e f dM_\alpha \vee b \int_S^e g dM_\alpha,$$

for all $a, b > 0$ and $f, g \in L_+^\alpha(S, \mu)$. For more details, see Stoev and Taqqu [35].

Now, for any collection of deterministic functions $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$, one can construct the stochastic process:

$$X_t = \int_S^e f_t(u) M_\alpha(du), \forall t \in T. \quad (2.2)$$

In view of the max-linearity of the extremal integrals and (2.1), the resulting process $X = \{X_t\}_{t \in T}$ is α -Fréchet. Furthermore, for any $n \in \mathbb{N}$, $x_i > 0$, $t_i \in T$, $1 \leq i \leq n$:

$$\mathbb{P}\{X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n\} = \exp \left\{ - \int_S \left(\bigvee_{1 \leq i \leq n} x_i^{-1} f_{t_i}(u) \right)^\alpha \mu(du) \right\}. \quad (2.3)$$

This shows that the deterministic functions $\{f_t\}_{t \in T}$ characterize completely the finite-dimensional distributions of the process $\{X_t\}_{t \in T}$. In general, if

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{ \int_S^e f_t dM_\alpha \right\}_{t \in T}, \quad (2.4)$$

for some $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$, we shall say that the process $X = \{X_t\}_{t \in T}$ has the *extremal integral* or *spectral representation* $\{f_t\}_{t \in T}$ over the space $L_+^\alpha(S, \mu)$. The f_t 's in (2.4) are also referred to as *spectral functions* of X .

Our goal in this paper is to characterize α -Fréchet processes in terms of their spectral representations. Many α -Fréchet processes of practical interest have tractable spectral representations. As shown in the proposition below, an α -Fréchet process X has the representation (2.4), where (S, μ) is a *standard Lebesgue space* (see Appendix A in [23]), if and only if, X satisfies *Condition S*.

Definition 2.2. An α -Fréchet process $X = \{X_t\}_{t \in T}$ is said to satisfy *Condition S* if there exists a countable subset $T_0 \subseteq T$ such that for every $t \in T$, we have that $X_{t_n} \xrightarrow{P} X_t$ for some $\{t_n\}_{n \in \mathbb{N}} \subset T_0$.

Proposition 2.1. An α -Fréchet process $X = \{X_t\}_{t \in T}$ has the extremal integral representation (2.4), with any (some) standard Lebesgue space (S, μ) and an α -Fréchet random sup-measure on S with control measure μ , if (only if) it satisfies Condition S.

The result above follows from Proposition 3.2 in [35], since the standard Lebesgue space (S, μ) may be chosen to be $[0, 1]$, equipped with the Lebesgue measure.

Remark 2.1. As shown in Kabluchko [15] (Theorem 1), every max-stable process can have a spectral representation over a sufficiently rich abstract measure space.

In the sequel, we focus only on the rich class of α -Fréchet processes that satisfy Condition S. This includes, for example, all measurable max-stable processes $X = \{X_t\}_{t \in T}$, indexed by a separable metric space T (see Proposition 5.2 below).

The fact that (S, μ) is a standard Lebesgue space implies that the space of integrands $L_+^\alpha(S, \mu)$ is a complete and *separable* metric space with respect to the metric:

$$\rho_{\mu, \alpha}(f, g) = \int_S |f^\alpha - g^\alpha| d\mu. \quad (2.5)$$

This metric is natural to use when handling extremal integrals, since as $n \rightarrow \infty$,

$$\int_S f_n dM_\alpha \xrightarrow{P} \xi, \quad \text{if and only if,} \quad \rho_{\mu, \alpha}(f_n, f) = \int_S |f_n^\alpha - f^\alpha| d\mu \rightarrow 0, \quad (2.6)$$

where $\xi = \int_S f dM_\alpha$ (see e.g. [35] and also Davis and Resnick [5]). In the sequel, we equip the space $L_+^\alpha(S, \mu)$ with the metric $\rho_{\mu, \alpha}$ and often write $\|f\|_{L_+^\alpha(S, \mu)}^\alpha$ for $\int_S f^\alpha d\mu$.

3 Max-Linear Isometries

The max-linear (sub)spaces of functions in $L_+^\alpha(S, \mu)$ play a key role in the representation and characterization of max-stable processes. We say that \mathcal{F} is a *max-linear sub-space* of $L_+^\alpha(S, \mu)$ if the following conditions hold:

- (i) $af \vee bg \in \mathcal{F}$, for all $a, b > 0, f, g \in \mathcal{F}$.
- (ii) $\mathcal{F} \subset L_+^\alpha(S, \mu)$ is closed in the metric $\rho_{\mu, \alpha}$.

In particular, we will frequently encounter the max-linear space $\mathcal{F} := \overline{\nabla\text{-span}}(f_t, t \in T)$, which is generated by the max-linear combinations $\bigvee_{1 \leq i \leq n} a_i f_{t_i}$, $t_i \in T$, $a_i > 0$, of the spectral functions in (2.4). In view of (2.6), the set of extremal integrals $\{\int_S f dM_\alpha, f \in \mathcal{F}\}$ is the smallest set that is closed with respect to convergence in probability and contains all max-linear combinations $\bigvee_{1 \leq i \leq n} a_i X_{t_i}$. For more details, see [35].

An α -Fréchet process $X = \{X_t\}_{t \in T}$ as in (2.2) has many equivalent spectral representations. They are all related, however, through *max-linear isometries* (see e.g. (4.1) below):

Definition 3.1. Let $\alpha > 0$. The map $U : L_+^\alpha(S_1, \mu_1) \rightarrow L_+^\alpha(S_2, \mu_2)$, is said to be a max-linear isometry, if:

- (i) $U(a_1 f_1 \vee a_2 f_1) = a_1(U f_1) \vee a_2(U f_2)$, μ_2 -a.e., for all $f_1, f_2 \in L_+^\alpha(S_1, \mu_1)$ and $a_1, a_2 \geq 0$.
- (ii) $\|U f\|_{L_+^\alpha(\mu_2)} = \|f\|_{L_+^\alpha(\mu_1)}$, for all $f \in L_+^\alpha(S_1, \mu_1)$.

The max-linear isometry U is called *max-linear isomorphism* if it is onto.

Consider a max-linear sub-space $\mathcal{F} \subset L_+^\alpha(S_1, \mu_1)$ and a max-linear isometry $U : \mathcal{F} \rightarrow L_+^\alpha(S_2, \mu_2)$. Our goal in this section is somewhat technical. Namely, to characterize U and also identify the largest max-linear sub-space $\mathcal{G} \subset L_+^\alpha(S_1, \mu_1)$, such that $\mathcal{F} \subset \mathcal{G}$ and U extends to \mathcal{G} uniquely as a max-linear isometry. This is done in Theorem 3.2 below. The proofs for all results in this section are given in Appendix A.1.

It is known that all linear isometries on L^α spaces for $\alpha \neq 2$ are related to a *regular set isomorphism* (see [19]). Regular set isomorphisms also play an important in the study of max-linear isometries.

Definition 3.2. Let $(S_1, \mathcal{S}_1, \mu_1)$ and $(S_2, \mathcal{S}_2, \mu_2)$ be two measure spaces. A set-mapping $T : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is said to be a *regular set isomorphism* if:

- (i) For all $A \in \mathcal{S}_1$, $T(S_1 \setminus A) = T(S_1) \setminus T(A) \mod \mu_2$;
- (ii) For disjoint A_n 's in \mathcal{S}_1 , $T(\cup_{n=1}^{+\infty} A_n) = \cup_{n=1}^{+\infty} T(A_n) \mod \mu_2$;
- (iii) $\mu_2(T(A)) = 0$ if and only if $\mu_1(A) = 0$.

Remark 3.1. Regular set isomorphisms are mappings defined modulo null sets. In the sequel, we often identify measurable sets that are equal modulo null sets.

The next properties follow immediately from the above definition:

- (iv) If $A_1, A_2 \in \mathcal{S}_1$ and $\mu_1(A_1 \cap A_2) = 0$, then $\mu_2(T(A_1) \cap T(A_2)) = 0$.
- (v) For all, not necessarily disjoint, $A_n \in \mathcal{S}_1$, $n \in \mathbb{N}$, we have:

$$T(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} T(A_n) \quad \text{and} \quad T(\cap_{n=1}^{\infty} A_n) = \cap_{n=1}^{\infty} T(A_n).$$

Any regular set isomorphism T induces a canonical function mapping Tf , defined for all measurable functions f , and such that $\{Tf \in B\} = T\{f \in B\}$, $\mod \mu_2$, for all Borel sets $B \in \mathcal{B}_{\mathbb{R}}$. The resulting mapping is linear and also max-linear. If T is, in addition, measure preserving, then the induced mapping becomes a max-linear isometry. For more details, see Lemma A.1 in Appendix A.1 or Doob [9]. The next result shows that any max-linear isometry, which maps the identity function $\mathbf{1}$ to the identity function $\mathbf{1}$, is induced by a measure preserving regular set isomorphism.

Theorem 3.1. Suppose $\alpha > 0$. Let \mathcal{F} be a max-linear sub-space of $L_+^\alpha(S_1, \mu_1)$ and $U : \mathcal{F} \rightarrow L_+^\alpha(S_2, \mu_2)$ be a max-linear isometry. If $\mathbf{1}_{S_1} \in \mathcal{F}$ and $U\mathbf{1}_{S_1} = \mathbf{1}_{S_2}$, then $Uf = Tf$ for all $f \in \mathcal{F}$, where:

- (i) T is induced by a measure preserving regular set isomorphism from $\sigma(\mathcal{F})$ onto $\sigma(U(\mathcal{F}))$,
- (ii) T is a max-linear isometry from $L_+^\alpha(S_1, \sigma(\mathcal{F}), \mu_1)$ onto $L_+^\alpha(S_2, \sigma(U(\mathcal{F})), \mu_2)$, and
- (iii) T is the unique extension of U to a max-linear isometry from $L_+^\alpha(S_1, \sigma(\mathcal{F}), \mu_1)$ to $L_+^\alpha(S_2, \mu_2)$.

Not all max-linear isometries are directly induced by regular set isomorphisms. We will show next, however, that every max-linear isometry can be related to a regular set isomorphism.

Definition 3.3. Let F be a collection of functions in $L_+^\alpha(S, \mu)$.

- (i) The *ratio σ -field* of F , written $\rho(F) := \sigma(\{f_1/f_2, f_1, f_2 \in F\})$, is defined as the σ -field generated by ratio of functions in F , where the ratios take values in the extended interval $[0, \infty]$;
- (ii) The *positive ratio space* of F , written $\mathcal{R}_+(F)$, is defined as $L_+^\alpha(S, \rho(F), \mu)$.
- (iii) The *extended positive ratio space* of F , written $\mathcal{R}_{e,+}(F)$, is defined as the class of all functions in $L_+^\alpha(S, \mu)$ that have the form rf , where r is non-negative $\rho(F)$ -measurable and $f \in F$.

In the following lemma, we present some important properties of the ratio σ -fields.

Lemma 3.1. *For any non-empty class of functions $F \subset L_+^\alpha(S, \mu)$, we have $\rho(F) = \rho(\overline{\text{span}}(F)) \subset \sigma(F)$. If, in addition, $\mathbf{1}_S \in F$, then $\rho(F) = \sigma(F)$.*

Before introducing the main result of this section, we need some auxiliary results about the notion of *full support*.

Definition 3.4. Let (S, μ) be a measurable space and F be a collection of measurable real-valued functions on (S, μ) . A measurable function f_0 is said to have *full support* w.r.t. F if $\mu(\text{supp}(g) \setminus \text{supp}(f_0)) = 0$ for all $g \in F$, where $\text{supp}(f) := \{f \neq 0\}$. If, in addition, $f_0 \in F$, we then write $\text{supp}(F) = \text{supp}(f_0)$.

Remark 3.2. Note that the definition of full support is modulo μ -null sets and the definition of $\text{supp}(F)$ is independent of the choice of $f_0 \in F$. Also, our definition of $\text{supp}(F)$ requires implicitly that F contains a function f_0 of full support.

Lemma 3.2. *Let \mathcal{F} be a max-linear sub-space of $L_+^\alpha(S, \mu)$. If \mathcal{F} is separable or μ is σ -finite, then there exists a function of full support in \mathcal{F} .*

Lemma 3.3. *Let \mathcal{F} be a max-linear sub-space of $L_+^\alpha(S_1, \mu_1)$ and let $U : \mathcal{F} \rightarrow L_+^\alpha(S_2, \mu_2)$ be a max-linear isometry. Assume that the measures μ_1 and μ_2 are σ -finite. If f_0 has full support in \mathcal{F} , then $U f_0$ has full support in $U(\mathcal{F})$.*

We now present the main result of this section.

Theorem 3.2. *Suppose $\alpha > 0$ and let \mathcal{F} be a max-linear sub-space of $L_+^\alpha(S_1, \mu_1)$. Suppose also that $\text{supp}(\mathcal{F}) = S_1$. If μ_1 is σ -finite and $U : \mathcal{F} \rightarrow L_+^\alpha(S_2, \mu_2)$ is a max-linear isometry, then:*

(i) *U has a unique extension to a max-linear isometry \overline{U} , defined on $\mathcal{R}_{e,+}(\mathcal{F})$ to $L_+^\alpha(S_2, \mu_2)$. Moreover, \overline{U} is also onto $\mathcal{R}_{e,+}(U(\mathcal{F})) \subset L_+^\alpha(S_2, \mu_2)$ and*

$$\overline{U}(rf) = (Tr)(Uf), \quad \text{for all } r \in \mathcal{R}_+(\mathcal{F}), f \in \mathcal{F}, \quad (3.1)$$

where the function mapping $T : \mathcal{R}_+(\mathcal{F}) \rightarrow \mathcal{R}_+(U(\mathcal{F}))$ is induced by a regular set isomorphism of $\rho(\mathcal{F})$ onto $\rho(U(\mathcal{F}))$.

(ii) *For all $f \in \mathcal{F}$, we have*

$$(Uf)^\alpha d\mu_2 = d\mu_{1,f} \circ T^{-1}, \quad (3.2)$$

where $d\mu_{1,f} = f^\alpha d\mu_1$.

Remark 3.3. Equality (3.2) means that the two measures are identical on the σ -field $\rho(U(F))$, i.e. $\int_A (Uf)^\alpha d\mu_2 = \mu_{1,f} \circ T^{-1}(A)$, for all $A \in \rho(U(F))$. In the sequel, we will interpret equalities between measures defined on different σ -fields as equality of their corresponding restrictions to the largest common σ -field. Note that in general $(Uf)^\alpha$ in (3.2) does not necessarily equal the Radon-Nikodym derivative $d(\mu_{1,f} \circ T^{-1})/d\mu_2$ since the σ -field $\rho(U(F))$ is typically rougher than \mathcal{B}_{S_2} . This is why U may not have a unique extension to $L_+^\alpha(S_2, \mu_2)$, in general. See Remark 3.2(c) in Rosiński [29] for a detailed discussion.

Recall the notion of equivalence in measure of two σ -fields, defined on the same measure space (S, \mathcal{S}, μ) . Namely, for two σ -fields $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$, we write $\mathcal{A} \sim \mathcal{B} \pmod{\mu}$, if for any $A \in \mathcal{A}$ ($B \in \mathcal{B}$, respectively), there exists $B \in \mathcal{B}$ ($A \in \mathcal{A}$, respectively) such that $\mu(A \Delta B) = 0$. The following result will be used in the next section.

Lemma 3.4. *Let F be a class of functions in $L_+^\alpha(S, \mu)$. Suppose there exists $f_0 \in F$ with full support in F . If $S = \text{supp}(f_0) \equiv \text{supp}(F)$ and if $\rho(F) \sim \mathcal{B}_S \pmod{\mu}$, then $\mathcal{R}_{e,+}(F) = L_+^\alpha(S, \mu)$.*

This result and Theorem 3.2, provide sufficient conditions for a max-linear isometry U , defined on F , to extend uniquely to the entire space $L_+^\alpha(S, \mu)$.

4 Minimal Representations for α -Fréchet Processes

Let $\{f_t^{(i)}\}_{t \in T} \subset L_+^\alpha(S_i, \mu_i)$, $i = 1, 2$ be two spectral representations for the α -Fréchet process $X = \{X_t\}_{t \in T}$. Recall that for all $t_j \in \mathbb{R}$, $c_j \geq 0$, $1 \leq j \leq n$, we have

$$\mathbb{P}\{X_{t_j} \leq c_j^{-1}, 1 \leq j \leq n\} = \int_{S_1} \left(\bigvee_{j=1}^n c_j f_{t_j}^{(1)} \right)^\alpha d\mu_1 = \int_{S_2} \left(\bigvee_{j=1}^n c_j f_{t_j}^{(2)} \right)^\alpha d\mu_2.$$

One can thus define the following natural max-linear isometry:

$$U : \overline{\nabla\text{-span}}\{f_t^{(1)}\}_{t \in T} \rightarrow \overline{\nabla\text{-span}}\{f_t^{(2)}\}_{t \in T}, \quad \text{with } U f_t^{(1)} := f_t^{(2)}, \quad \text{for all } t \in T. \quad (4.1)$$

In the sequel, U will be called the *relating max-linear isometry* of the two representations. Our goal in this section is to provide convenient representations for the max-linear isometry U .

For any standard Lebesgue space (S, μ) , we have that $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$ is separable, and hence by Lemma 3.2, the max-linear space $\mathcal{F} = \overline{\nabla\text{-span}}(f_t, t \in T)$ contains a function with *full support*. Therefore, by convention, we define the support of $\{f_t\}_{t \in T}$ as follows:

$$\text{supp}\{f_t, t \in T\} := \text{supp}(\mathcal{F}) \equiv \text{supp}\left(\overline{\nabla\text{-span}}(f_t, t \in T)\right).$$

In view of Theorem 3.2, one can readily represent the max-linear isometry U in (4.1) in terms of a regular set isomorphism. The latter mapping however is a set-mapping rather than point mapping. It is desirable to be able to express U via measurable point mappings. Unfortunately, in general such point mappings may not be unique. In order to have a unique point mapping relating the two representations, we need to impose further *minimality condition* on the spectral representations. The following definition is as in Rosiński [27] (see also [13]).

Definition 4.1. A spectral representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$ of an α -Fréchet process is said to be *minimal* if:

- (i) $\text{supp}\{f_t : t \in T\} = S$ μ -a.e., and
- (ii) for any $B \in \mathcal{B}_S$, there exists $A \in \rho(\{f_t : t \in T\})$ such that $\mu(A \Delta B) = 0$.

We shall also consider minimal representations with *standardized support* defined as follows.

Definition 4.2. A minimal representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$ has *standardized support* if, up to μ -null sets:

- (i) $S \subset (0, 1) \cup \mathbb{N}$,
- (ii) $S \cap (0, 1) = \emptyset$ or $(0, 1)$ and $\mu|_{(0,1)}$ is the Lebesgue measure,
- (iii) $S \cap \mathbb{N} = \emptyset, \mathbb{N}$ or $\{1, \dots, N\}$, where $N \in \mathbb{N}$ and $\mu|_{S \cap \mathbb{N}}$ is the counting measure.

Let $(S_{I,N}, \lambda_{I,N})$ denote the standard support with $I = 0$ or 1 respectively according to the two cases in (i) and $N = 0, N = \infty$ or $N \in \mathbb{N}$ respectively according to the three cases in (ii), e.g. $S_{0,\infty} = \mathbb{N}$ and $S_{1,N} = (0, 1) \cup \{1, \dots, N\}$.

We now show that any spectral representation of an α -Fréchet process can be transformed into a minimal one with standardized support.

Theorem 4.1. *Every α -Fréchet process satisfying Condition S has a minimal representation $\{f_t\}_{t \in T}$ with standardized support $(S_{I,N}, \lambda_{I,N})$. That is*

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{ \int_{S_{I,N}}^e f_t(s) M_\alpha(ds) \right\}_{t \in T}, \quad (4.2)$$

where M_α is the α -Fréchet random sup-measure with control measure $\lambda_{I,N}$.

Proof. By Proposition 2.1, one can let $G = \{g_t\}_{t \in T} \subset L_+^\alpha((0,1), \mathcal{B}_{(0,1)}, ds)$ be a spectral representation of the process in question, where ds is the Lebesgue measure on $(0,1)$. First, we study the ratio σ -field generated by G . Let $\mathcal{G} = \nabla\text{-span}\{g_t, t \in T\}$ and, in view of Lemma 3.2, let $g \in \mathcal{G}$ have full support in \mathcal{G} . By Lemma 3.1, we have $\rho(G) = \rho(\mathcal{G})$. Without loss of generality we assume $\text{supp}(g) = \text{supp}(\mathcal{G}) = (0,1)$ and $\|g\|_\alpha = 1$. Define a new measure μ on the space $((0,1), \rho(\mathcal{G}))$ by setting $d\mu(s) = g(s)^\alpha ds$. Since μ is a probability measure, the measure space $((0,1), \rho(\mathcal{G}), \mu)$ has at most countably many (equivalence classes of) atoms. With some abuse of notation, we represent them as A_1, A_2, \dots, A_N , where $N = 0$ means no atoms, $N \in \mathbb{N}$ for finite number of atoms, and $N = \infty$ when countably infinite number of atoms are present. Set $A = \bigcup_{n=1}^N A_n$ and $a_i = \mu(A_i)$, $1 \leq i \leq N$.

Next, we define a regular set isomorphism T_r of measure space $((0,1), \rho(\mathcal{G}), \mu)$ onto measure space $(S_{I,N}, \mathcal{B}_{S_{I,N}}, \lambda_{I,N})$ considered in Definition 4.2. For the atoms, define $T_r^N(A_n) = \{n\}$, $n \leq N$, $n \in \mathbb{N}$. For the non-atomic subset $A_0 \equiv (0,1) \setminus A$, let $\mathcal{S}_0 = \rho(\mathcal{G}) \cap A_0 = \{B \cap A_0, B \in \rho(\mathcal{G})\}$ and let μ_i be the restriction of μ to A_i , $i = 0, \dots, N$.

The case $a_0 = 0$ is trivial since then $\mu(A_0) = \mu_0(A_0) = 0$ and we can simply ignore $(A_0, \mathcal{S}_0, \mu_0)$. We thus suppose that $a_0 > 0$ and observe that $(A_0, \mathcal{S}_0, \mu_0)$ is a non-atomic separable measurable space (see p167 in [11]) with total mass $\mu(A_0) = 1 - \sum_{n=1}^N a_n \equiv a_0$. Indeed, the separability of $(A_0, \mathcal{S}_0, \mu_0)$ is due to the fact that \mathcal{G} restricted on A_0 is separable.

Now, Theorem 41.C in Halmos [11] implies that there is a measure preserving regular set isomorphism, i.e., a *measure algebra isomorphism* T_r^I from $(A_0, \mathcal{S}_0, \mu_0)$ onto $((0,1), \mathcal{B}_{(0,1)}, a_0 ds)$. By combining the definitions of T_r^N on all atoms A_i , $1 \leq i \leq N$ and T_r^I on $(A_0, \mathcal{S}_0, \mu_0)$, we thus obtain a regular set isomorphism $T_r := T_r^I + T_r^N$ from $((0,1), \rho(\mathcal{G}), \mu)$ onto $(S_{I,N}, \mathcal{B}_{S_{I,N}}, \lambda_{I,N})$. Note that T_r is not necessarily measure preserving.

By using T_r , we construct next the desired minimal representation with standardized support. Define

$$f_t(s) = T_r(g_t/g)(s) \left(a_0^{1/\alpha} \mathbf{1}_{(0,1)}(s) + \sum_{n=1}^N a_n^{1/\alpha} \mathbf{1}_{\{n\}}(s) \right), \quad (4.3)$$

where T_r is the canonical map on measurable functions induced by the constructed isomorphism (see Lemma A.1 or p452-454 [9]) from $L_+^\alpha((0,1), \rho(\mathcal{G}), \mu)$ onto $L_+^\alpha(S_{I,N}, \lambda_{I,N})$. We claim that $\{f_t\}_{t \in T}$ is a minimal representation with standardized support. It is

clearly a spectral representation, since, for any $m \in \mathbb{N}, t_i \in T, c_i > 0, 1 \leq i \leq m$,

$$\begin{aligned}
\left\| \bigvee_{i=1}^m c_i f_{t_i} \right\|_{L_+^\alpha(S_{I,N}, \lambda_{I,N})}^\alpha &= \left\| \bigvee_{i=1}^m c_i T_r(g_{t_i}/g) \left(a_0^{1/\alpha} \mathbf{1}_{(0,1)} + \sum_{n=1}^N a_n^{1/\alpha} \mathbf{1}_{\{n\}} \right) \right\|_{L_+^\alpha(S_{I,N}, \lambda_{I,N})}^\alpha \\
&= \left\| a_0^{1/\alpha} \bigvee_{i=1}^m c_i T_r(g_{t_i}/g) \right\|_{L_+^\alpha(0,1)}^\alpha + \sum_{n=1}^N \left| a_n^{1/\alpha} \bigvee_{i=1}^m c_i T_r(g_{t_i}/g)(n) \right|^\alpha \\
&= \left\| \bigvee_{i=1}^m c_i g_{t_i}/g \right\|_{L^\alpha(A_0, \mu_0)}^\alpha + \sum_{n=1}^N \left\| \bigvee_{i=1}^m c_i g_{t_i}/g \right\|_{L^\alpha(A_n, \mu_n)}^\alpha \quad (4.4) \\
&= \left\| \bigvee_{i=1}^m c_i g_{t_i}/g \right\|_{L^\alpha((0,1), \mu)}^\alpha = \left\| \bigvee_{i=1}^m c_i g_{t_i} \right\|_{L_+^\alpha(0,1)}^\alpha
\end{aligned}$$

where (4.4) follows from the fact that T_r^I is a measure preserving regular set isomorphism of A_0 onto $(0,1)$ and since T_r^N maps atoms to integer points in a one-to-one and onto manner. Indeed, restricted on each $A_i, 0 \leq i \leq N$, $a_i^{1/\alpha} T_r$ is a max-linear isometry satisfying

$$\begin{aligned}
\left\| a_i^{1/\alpha} T_r \mathbf{1}_{A_i} \right\|_{L^\alpha(T_r(A_i), \lambda_{I,N})}^\alpha &= \left\| a_i^{1/\alpha} \mathbf{1}_{T_r A_i} \right\|_{L^\alpha(T_r(A_i), \lambda_{I,N})}^\alpha \\
&= a_i \lambda_{I,N}(T_r A_i) = \mu(A_i) = \|\mathbf{1}_{A_i}\|_{L^\alpha(A_i, \mu_i)}^\alpha.
\end{aligned}$$

We will complete the proof by verifying the minimality of $\{f_t\}_{t \in T}$ (by Definition 4.1). Let \mathcal{F} denote $\overline{\nabla\text{-span}}\{f_t, t \in T\}$ and note that $g \in \mathcal{G} = \overline{\nabla\text{-span}}\{g_t, t \in T\}$. Since $T_r(g/g) = \mathbf{1}_{S_{I,N}}$, by (4.3), we obtain that

$$f_{I,N}(s) := a_0^{1/\alpha} \mathbf{1}_{(0,1)}(s) + \sum_{n=1}^N a_n^{1/\alpha} \mathbf{1}_{\{n\}}(s) \text{ belongs to } \mathcal{F}. \quad (4.5)$$

This implies $\text{supp}(f_{I,N}) = \text{supp}(\mathcal{F}) = S_{I,N}$, and whence (i) in Definition 4.1 holds. To verify (ii), observe that by (4.3) and Lemma A.1, $f_1/f_2 = T_r(g_1/g)/T_r(g_2/g) = T_r(g_1/g_2)$ for all $g_1, g_2 \in \mathcal{G}$. Therefore $T_r(\rho(\mathcal{G})) \equiv \rho(\mathcal{F})$, and since, as shown above, the regular set isomorphism T_r maps $\rho(\mathcal{G})$ onto $\mathcal{B}_{S_{I,N}}$, it follows that (ii) holds. \square

Remark 4.1. Theorem 4.1 shows the existence of minimal representations with standardized support. One can have many minimal representations whose supports are not necessarily standardized in the same way. For example, in the proof of Theorem 4.1, we could define $\tilde{\lambda}_{I,N}$ on $S_{I,N}$ so that restricted on the atoms $A_i, 1 \leq i \leq N$, we have $d\tilde{\lambda}_{I,N} = a_i^{1/\alpha} d\lambda_{I,N}$. In this case, one obtains a finite measure $\tilde{\lambda}_{I,N}$ on $S_{I,N}$ as discussed in Rosiński [26] (p. 626) for the case of symmetric α -stable processes. Our measure $\lambda_{I,N}$ may be infinite, since it is a counting measure on the atoms.

Remark 4.2. Theorem 4.1 can be seen as a generalization of Theorem 4.1 in de Haan and Pickands III [8]. Instead of minimal representation, *proper representation* is involved therein. A spectral representation is proper if the spectral functions $\{f_t\}_{t \in T}$ satisfy (i) $\text{supp}\{f_t, t \in T\} = S, \mu$ -a.e. and (ii) $\forall B \in \mathcal{B}_S$, either there exists $A \in \rho(\{f_t, t \in T\})$ such that $\mu(A \Delta B) = 0$ or there exists an atom $A \in \rho(\{f_t, t \in T\})$ such that $\mu(B \cap A) > 0$. This definition is closely related to our definition of minimality, in the sense that any proper representation can be transformed into a minimal one. Indeed, this essentially involves contracting the atoms to points as in the proof of Theorem 4.1.

Consider the *canonical* max-linear isometry U relating two spectral representations as in (4.1). Theorem 3.2 implies that U extends uniquely to a max-linear isometry $U : \mathcal{R}_{e,+}(\mathcal{F}^{(1)}) \rightarrow \mathcal{R}_{e,+}(\mathcal{F}^{(2)})$ between extended positive ratio spaces, where $\mathcal{F}^{(i)} = \overline{\nabla\text{-span}}\{f_t^{(i)} : t \in T\}$, $i = 1, 2$. Now, if the first spectral representation $\{f_t^{(1)}\}_{t \in T}$ is *minimal*, then by Lemma 3.4, $\mathcal{R}_{e,+}(\mathcal{F}^{(1)}) = L_+^\alpha(S_{I,N}, \lambda_{I,N})$. In this case, one can also represent U in terms of *measurable point mappings*. This *point mapping representation* is developed in the following result. It will be essential for our studies in Sections 5 and 6.

Theorem 4.2. *Let $\{f_t\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$ and $\{g_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$ be two spectral representations of an α -Fréchet process $\{X_t\}_{t \in T}$. Let U be the relating max-linear isometry of $\{f_t\}_{t \in T}$ and $\{g_t\}_{t \in T}$. If $\{f_t\}_{t \in T}$ is minimal and $\{g_t\}_{t \in T}$ is arbitrary, then*

- (i) *U can be uniquely extended to $L_+^\alpha(S_{I,N}, \lambda_{I,N})$;*
- (ii) *U can be represented by measurable functions $\Phi : S \rightarrow S_{I,N}$ and $h : S \rightarrow \mathbb{R}_+ \setminus \{0\}$, such that Φ is onto, and the following statements hold:*

$$g_t(s) = U f_t(s) = h(s) (f_t \circ \Phi)(s), \quad \mu\text{-a.e.}, \quad (4.6)$$

and

$$d\lambda_{I,N} = d(\mu_h \circ \Phi^{-1}), \quad (4.7)$$

where $d\mu_h(s) = h(s)^\alpha d\mu$. Φ is unique modulo μ .

Proof. Let F and G denote $\{f_t\}_{t \in T}$ and $\{g_t\}_{t \in T}$ respectively. By Theorem 3.2, there exists a regular set isomorphism T_r from $\mathcal{B}_{S_{I,N}}$ onto $\rho(G)$ such that

$$g_t(s) = U f_t(s) = (T_r f_t)(s) \left(\frac{U f_0}{T_r f_0} \right)(s), \quad \mu\text{-a.e.}, \forall t \in T,$$

for some function with full support $f_0 \in \overline{\nabla\text{-span}}\{f_t, t \in T\}$. In the last relation we used the facts that $T_r(1/f_0) = 1/T_r(f_0)$ and $T_r(f_t/f_0) = T_r(f_t)/T_r(f_0)$ (Lemma A.1). Moreover, we have that

$$(U f_0)^\alpha d\mu = d(\mu_{1,f_0} \circ T_r^{-1}) = (T_r f_0)^\alpha d(\lambda_{I,N} \circ T_r^{-1}), \quad \mu\text{-a.e.} \quad (4.8)$$

By Theorem 32.5 in Sikorski [33], the regular set isomorphism T_r can be induced by a point mapping Φ from S onto $S_{I,N}$ such that $T_r f = f \circ \Phi$, for all measurable functions f defined on $S_{I,N}$. Moreover, Φ is unique modulo μ . Note that in general Φ is not one-to-one, because of the possible presence of atoms in $(S, \rho(\mathcal{G}), \mu)$. To show that (4.7) is true, let

$$\tilde{h}(s) = \frac{U f_0}{T_r f_0}(s) = \frac{U f_0}{f_0 \circ \Phi}(s).$$

Note that by Lemma 3.3, $\tilde{h}(s) > 0$, μ -a.e.. Put

$$h(s) = \begin{cases} \tilde{h}(s) & \text{if } \tilde{h}(s) > 0 \\ 1 & \text{if } \tilde{h}(s) = 0 \end{cases} \quad \text{and } d\mu_h = h^\alpha d\mu. \quad (4.9)$$

Observe that h is a measurable function from S to $\mathbb{R}_+ \setminus \{0\}$. Thus, relation (4.7) follows by (4.9) and (4.8). This completes the proof. \square

Remark 4.3. Relation (4.7) and the fact that $h(s) > 0$ for all s imply that $\mu \circ \Phi^{-1} \sim \lambda_{I,N}$.

Now, if both representations in Theorem 4.2 are minimal, we have the following:

Corollary 4.1. *If $\{f_t^{(i)}\}_{t \in T}, i = 1, 2$ are two minimal representations of an α -Fréchet process $\{X_t\}_{t \in T}$ with standardized support $(S_{I_i, N_i}, \lambda_{I_i, N_i}), i = 1, 2$, then the relating max-linear isometry U from $L_+^\alpha(S_{I_1, N_1}, \lambda_{I_1, N_1})$ onto $L_+^\alpha(S_{I_2, N_2}, \lambda_{I_2, N_2})$ is determined by, unique modulo λ_{I_2, N_2} , functions $\Phi : S_{I_2, N_2} \rightarrow S_{I_1, N_1}$ and $h : S_{I_2, N_2} \rightarrow \mathbb{R}_+ \setminus \{0\}$ such that Φ is one-to-one and onto and, for each $t \in T$,*

$$f_t^{(2)}(s) = U f_t^{(1)}(s) = h(s) \left(f_t^{(1)} \circ \Phi \right)(s), \quad \lambda_{I_2, N_2}\text{-a.e.} \quad (4.10)$$

and

$$\frac{d(\lambda_{I_1, N_1} \circ \Phi)}{d\lambda_{I_2, N_2}}(s) = h(s)^\alpha, \quad \lambda_{I_2, N_2}\text{-a.e.} \quad (4.11)$$

An important consequence of Corollary 4.1 is the following.

Corollary 4.2. *Let $\{f_t^{(i)}\}_{t \in T}, i = 1, 2$ be as in Corollary 4.1. Then*

$$I_1 = I_2 = I \quad \text{and} \quad N_1 = N_2 = N.$$

Moreover, the relating max-linear isometry $U : L_+^\alpha(S_{I, N}, \lambda_{I, N}) \rightarrow L_+^\alpha(S_{I, N}, \lambda_{I, N})$ satisfies

(i) if $I = 1$, then $\forall f \in L_+^\alpha(0, 1)$,

$$Uf = \left(\frac{d(\lambda \circ \Phi_I)}{d\lambda} \right)^\alpha (f \circ \Phi_I), \lambda\text{-a.e.}, \quad (4.12)$$

where λ is the Lebesgue measure on $(0, 1)$, Φ_I is a point map from $(0, 1)$ onto $(0, 1)$, and

(ii) if $N \neq 0$, then $\forall f \in L_+^\alpha(S_{I, N} \cap \mathbb{N}, \lambda_{I, N})$,

$$Uf = f \circ \Phi_N, \quad (4.13)$$

where Φ_N is an automorphism of $S_{I, N} \cap \mathbb{N}$.

Proof. We start by recalling that U is induced by T_r , which is an one-to-one isomorphism modulo $\lambda_{I, N}$ -null sets from $\mathcal{B}_{S_{I_1, N_1}}$ onto $\mathcal{B}_{S_{I_2, N_2}}$ (by Theorem 3.2). Since T_r is a regular set isomorphism, one has that for all $A, B \in \mathcal{B}_{S_{I_1, N_1}}$,

$$\lambda_{I_1, N_1}(A) \lambda_{I_1, N_1}(B \setminus A) = 0 \Leftrightarrow \lambda_{I_2, N_2}(T_r A) \lambda_{I_2, N_2}(T_r B \setminus T_r A) = 0.$$

Thus T_r maps *atoms* to *atoms* and non-atomic sets to non-atomic sets. Hence,

$$T_r \left(\mathcal{B}_{S_{I_1, N_1}} \cap (0, 1) \right) \subset \mathcal{B}_{S_{I_2, N_2}} \cap (0, 1) \quad \text{and} \quad T_r \left(\mathcal{B}_{S_{I_1, N_1}} \cap \mathbb{N} \right) \subset \mathcal{B}_{S_{I_2, N_2}} \cap \mathbb{N}.$$

Since T_r is onto, we also have that

$$T_r \left(\mathcal{B}_{S_{I_1, N_1}} \cap (0, 1) \right) = \mathcal{B}_{S_{I_2, N_2}} \cap (0, 1) \quad \text{and} \quad T_r \left(\mathcal{B}_{S_{I_1, N_1}} \cap \mathbb{N} \right) = \mathcal{B}_{S_{I_2, N_2}} \cap \mathbb{N}.$$

This implies that $I_1 = I_2$. Moreover, since T_r is one-to-one and onto, we have $N_1 = N_2$. This also shows that $T_r : S_{I, N} \cap \mathbb{N} \rightarrow S_{I, N} \cap \mathbb{N}$ is a bijection where $I := I_1 = I_2$ and $N := N_1 = N_2$. By Corollary 4.1, it follows that (i) and (ii) holds. Note that in (ii) we have simpler formula for Uf . This is because that on the discrete part $S_{I, N} \cap \mathbb{N}$, the function $h(s)$ defined in (4.11) equals 1. \square

Remark 4.4. Theorem 4.2 and Corollary 4.1 are valid even if the minimal representations therein do not have standardized support (see Theorem 4.1 and Theorem 4.2 in [8] for results on discrete processes; see also Theorem 2.1 in Rosiński [27] for analogous result in the sum-stable setting). The advantage of having minimal representation with *standardized support* is shown in Corollary 4.2 and further exploited in the next section.

5 Classification of α -Fréchet Processes

We now apply the abstract results on max-linear isometries and minimal representations to classify α -Fréchet processes. The first classification result is an immediate consequence of the notion of minimal representation with standardized support and it applies to general max-stable processes.

5.1 Continuous-discrete decomposition

Consider an α -Fréchet process $X = \{X_t\}_{t \in T}$, which has a minimal representation with standardized support $\{f_t\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$. By Corollary 4.2, the support $(S_{I,N}, \lambda_{I,N})$ is unique. We therefore call $S_{I,N}$ the *standardized support* of X and focus on the *continuous* and *discrete* parts of $S_{I,N}$, respectively:

$$S_I := S_{I,N} \cap (0, 1), \quad \text{and} \quad S_N := S_{I,N} \cap \mathbb{N}.$$

Let $f_t^I = f_t \mathbf{1}_{S_I}$, and $f_t^N = f_t \mathbf{1}_{S_N}$ be the restrictions of the f_t 's to S_I and S_N , respectively. One can write:

$$\{X_t\}_{t \in T} \stackrel{d}{=} \{X_t^I \vee X_t^N\}_{t \in T}, \quad (5.1)$$

where

$$X_t^I := \int_{S_I}^e f_t^I(s) M_\alpha(ds) \quad \text{and} \quad X_t^N := \int_{S_N}^e f_t^N(s) M_\alpha(ds), \quad (5.2)$$

are two independent α -Fréchet processes. The following result shows that the decomposition (5.1) does not depend on the choice of the representation $\{f_t\}_{t \in T}$.

Theorem 5.1. *Let $\{X_t\}_{t \in T}$ be an α -Fréchet process with minimal representation of standardized support $\{f_t\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$. Then:*

- (i) *The decomposition (5.1) is unique in distribution.*
- (ii) *The processes $X^I = \{X_t^I\}_{t \in T}$ and $X^N = \{X_t^N\}_{t \in T}$ are independent and they have standardized supports S_I and S_N , respectively.*
- (iii) *The functions $\{f_t^I\}_{t \in T} \subset L_+^\alpha(S_I, \lambda_I)$ and $\{f_t^N\}_{t \in T} \subset L_+^\alpha(S_N, \lambda_N)$ provide minimal representations for the processes X^I and X^N , respectively.*

Proof. To prove (i), suppose $\{g_t\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$ is another minimal representation of X with standardized support and consider the decomposition $\{X_t\}_{t \in T} \stackrel{d}{=} \{Y_t^I \vee Y_t^N\}_{t \in T}$, where

$$Y_t^I := \int_{S_I}^e g_t^I(s) M_\alpha(ds) \quad \text{and} \quad Y_t^N := \int_{S_N}^e g_t^N(s) M_\alpha(ds), \quad \forall t \in T.$$

By Corollary 4.2, the relating max-linear isometry U of $\{f_t\}_{t \in T}$ and $\{g_t\}_{t \in T}$ is such that for all $t \in T$, $U(f_t^I) = g_t^I$ and $U(f_t^N) = g_t^N$. Moreover, U remains a max-linear isometry when restricted to S_I and S_N , and hence

$$\{X_t^I\}_{t \in T} \stackrel{d}{=} \{Y_t^I\}_{t \in T} \quad \text{and} \quad \{X_t^N\}_{t \in T} \stackrel{d}{=} \{Y_t^N\}_{t \in T}.$$

The last two relations imply that the decomposition (5.1) does not depend on the choice of the representation. The components $\{X_t^I\}_{t \in T}$ and $\{X_t^N\}_{t \in T}$ are independent since they are defined by extremal integrals over two disjoint sets S_I and S_N . The minimality of $\{f_t\}_{t \in T}$ implies the minimality of $\{f_t^I\}_{t \in T}$ and $\{f_t^N\}_{t \in T}$, restricted to S_I and S_N , respectively. This completes the proof, since the supports S_I and S_N of $\{f_t^I\}_{t \in T}$ and $\{f_t^N\}_{t \in T}$ are standardized (Definition 4.2). \square

The processes $\{X_t^I\}_{t \in T}$ and $\{X_t^N\}_{t \in T}$ in the Decomposition (5.1) will be referred to as the *spectrally continuous* and *spectrally discrete* components of X , respectively. The next result clarifies further their structure.

Corollary 5.1. *Let $\{f_t\}_{t \in T}$ and $\{g_t\}_{t \in T}$ be two minimal representations with standardized support of an α -Fréchet process $\{X_t\}_{t \in T}$. Then, the relating max-linear isometry U of these representations, has the form*

$$U f_t^I = \left(\frac{d(\lambda \circ \Phi_I)}{d\lambda} \right)^{1/\alpha} (f_t^I \circ \Phi_I) = g_t^I \quad \text{and} \quad U f_t^N = f_t^N \circ \Phi_N = g_t^N, \lambda\text{-a.e.}, \quad \forall t \in T, \quad (5.3)$$

where Φ_I is a point mapping from S_I onto S_I and Φ_N is a permutation of S_N (a one-to-one mapping from S_N onto S_N).

The proof is an immediate consequence of Relations (4.12) and (4.13) above. This result shows that the discrete component of an α -Fréchet process has an interesting invariance property. Namely, suppose that X has a non-trivial discrete component $X^N = \{X_t^N\}_{t \in T}$. By Corollary 5.1, there exists a *unique* set of functions $t \mapsto \phi_t(i)$, $i \in S_N$, $t \in T$, such that: (i) $\text{supp}\{\phi_t, t \in T\} \equiv S_N$, (ii) $\rho\{\phi_t, t \in T\} = \mathcal{B}_{S_N} \equiv 2^{S_N}$ and (iii) $\sum_{1 \leq i \leq N} \phi_t(i)^\alpha < \infty$, for all $t \in T$ and

$$\{X_t^N\}_{t \in T} \stackrel{d}{=} \bigvee_{i=1}^N \phi_t(i) Z_i,$$

where Z_i , $1 \leq i \leq N$ are independent standard α -Fréchet random variables. The functions $t \mapsto \phi_t(i)$, $1 \leq i \leq N$ do not depend on the particular representation of X^N . By analogy with the Karhunen-Loève decomposition of Gaussian processes (see e.g. p57 in [14]), we call the functions $t \mapsto \phi_t(i)$ the *discrete principal components* of X .

Proposition 5.1. *The finite or countable collection of functions $\{t \mapsto \phi_t(i), i \in S_N, t \in T\}$, $N \in \mathbb{N} \cup \{\infty\}$ can be the discrete principal components of an α -Fréchet process, if and only if, the representation $\{\phi_t\}_{t \in T} \subset L_+^\alpha(S_N, \lambda_N)$ is minimal.*

The proof is trivial. We state this result to emphasize that not every collection of non-negative functions can serve as discrete principal components. The *minimality* constraint can be viewed as the counterpart of the *orthogonality* condition on the principal components in the Gaussian case. The following two examples illustrate typical spectrally discrete and spectrally continuous processes.

Example 5.1. Let Z_i , $i \in \mathbb{N}$ be independent standard α -Fréchet variables and let $g_t(i) \geq 0$, $t \in T$ be such that $\sum_{i \in \mathbb{N}} g_t^\alpha(i) < \infty$, for all $t \in T$. It is easy to see that the α -Fréchet process

$$X_t := \bigvee_{i \in \mathbb{N}} g_t(i) Z_i \equiv \int_{\mathbb{N}}^e g_t dM_\alpha, \quad t \in T,$$

is *spectrally discrete*. That is, $X = \{X_t\}_{t \in T}$ has trivial spectrally continuous component. Indeed, this follows from Theorem 4.2 since the mapping Φ therein is onto, and thus the set $\Phi(\mathbb{N}) = S_{I,N}$ is necessarily countable.

Example 5.2. Consider the well-known α -Fréchet *extremal process* ($\alpha > 0$):

$$\{X_t\}_{t \in \mathbb{R}_+} \stackrel{d}{=} \left\{ \int_{\mathbb{R}_+}^e \mathbf{1}_{(0,t]}(u) M_\alpha(du) \right\}_{t \in \mathbb{R}_+}, \quad (5.4)$$

where M_α has the Lebesgue control measure on \mathbb{R}_+ . The process $X = \{X_t\}_{t \in \mathbb{R}_+}$ can be viewed as the max-stable counterpart to a sum-stable Lévy process. This is because X has *independent max-increments*, i.e., for any $0 = t_0 < t_1 < \dots < t_n$,

$$(X_{t_1}, \dots, X_{t_n}) \stackrel{d}{=} (\xi_1, \xi_1 \vee \xi_2, \dots, \xi_1 \vee \dots \vee \xi_n),$$

where $\xi_i = M_\alpha((t_{i-1}, t_i])$, $1 \leq i \leq n$. The representation in (5.4) is minimal but its support is not standardized. Let

$$f_t(s) := s^{-1/\alpha} \mathbf{1}_{(0,t]}(\log(1/s)), \quad s \in (0, 1),$$

and observe that $f_t(s) \in L_+^\alpha((0, 1), ds)$. By using a change of variables one can show that

$$\{X_t\}_{t \in \mathbb{R}_+} \stackrel{d}{=} \left\{ \int_{(0,1)}^e f_t(s) M_\alpha(ds) \right\}_{t \in \mathbb{R}_+},$$

where the last representation is minimal and has standardized support. Thus, the α -Fréchet extremal process X is *spectrally continuous*.

5.2 Classification via co-spectral functions

Here we present a characterization of α -Fréchet processes based on a different point of view. Namely, instead of focusing on the spectral functions $s \mapsto f_t(s)$, we now consider the *co-spectral functions* $t \mapsto f_t(s)$, which are functions of t , with s fixed. To be able to handle the co-spectral functions, we suppose that T is a *separable* metric space with respect to a metric ρ_T and let \mathcal{T} be its Borel σ -algebra. We say that the spectral representation $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S, \mu)$ is jointly measurable if the mapping $(t, s) \mapsto f_t(s)$ is measurable w.r.t. the product σ -algebra $\mathcal{T} \otimes \mathcal{S} := \sigma(\mathcal{T} \times \mathcal{S})$. The following result clarifies the connection between the joint measurability of the spectral functions $f_t(s)$ and the measurability of its corresponding α -Fréchet process.

Proposition 5.2. *Let (S, μ) be a standard Lebesgue space and M_α ($\alpha > 0$) be an α -Fréchet random sup-measure on S with control measure μ . As above, let (T, ρ_T) be a separable metric space.*

(i) *Let $X = \{X_t\}_{t \in T}$ have a spectral representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$ as in (2.4). Then, X has a measurable modification if and only if $\{f_t(s)\}_{t \in T}$ has a jointly measurable modification, i.e., there exists a $\mathcal{T} \otimes \mathcal{B}_S$ -measurable mapping $(s, t) \mapsto g_t(s)$, such that $f_t(s) = g_t(s)$ μ -a.e. for all $t \in T$.*

(ii) *If an α -Fréchet process $X = \{X_t\}_{t \in T}$ has a measurable modification, then it satisfies Condition S (see Definition 2.2), and hence it has a representation as in (2.4).*

The proof is given in Appendix. The above result shows that for a measurable α -Fréchet process $X = \{X_t\}_{t \in T}$, one can always have a representation as in (2.4), with jointly measurable spectral representations. Conversely, any X as in (2.4) with measurable spectral functions has a measurable modification.

Let now λ be a σ -finite Borel measure on T . We will view each $f.(s)$ as an element of the classes $L_+^0(T, \mathcal{T}, \lambda)$ of non-negative \mathcal{T} -measurable functions, identified with respect to equality λ -almost everywhere. Recall that a set $\mathcal{P} \subset L_+^0(T, \mathcal{T}, \lambda)$ is said to be a *positive cone* in $L_+^0(T, \mathcal{T}, \lambda)$, if $c\mathcal{P} \subset \mathcal{P}$ for all $c \geq 0$. Two cones \mathcal{P}_1 and \mathcal{P}_2 are *disjoint* if $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$.

We propose a general strategy for classification of α -Fréchet processes, based on any collection of disjoint positive cones $\mathcal{P}_j \subset L_+^0(T, \mathcal{T}, \lambda)$, $1 \leq j \leq n$. For any α -Fréchet process $X = \{X_t\}_{t \in T}$ with jointly measurable representation of full support $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S, \mu)$, we say the representation has a *co-spectral decomposition* w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$, if there exist measurable sets $S^{(j)}$, $1 \leq j \leq n$, such that

$$S^{(j)} \subset \{s \in S : f.(s) \in \mathcal{P}_j\}, \quad 1 \leq j \leq n \quad \text{and} \quad \mu\left(S \setminus \bigcup_{j=1}^n S^{(j)}\right) = 0. \quad (5.5)$$

The sets $S^{(j)}$, $1 \leq j \leq n$ are modulo μ disjoint. Indeed, Let $A := \{s \in S : f.(s) \equiv 0\}$ and note that $\mu(A) = 0$ by the fact $\text{supp}\{f_t, t \in T\} = S$ modulo μ and Fubini's Theorem. Since $\mathcal{P}_j \cap \mathcal{P}_k = \{0\}$, we have that $S^{(j)} \cap S^{(k)} = A$ for all $1 \leq j \neq k \leq n$. That is, the space S is partitioned into n modulo μ disjoint components:

$$S = S^{(1)} \cup \dots \cup S^{(n)} \quad \text{mod } \mu, \quad \text{with } \mu(S^{(j)} \cap S^{(k)}) = 0, \quad j \neq k. \quad (5.6)$$

This yields the *decomposition*:

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{X_t^{(1)} \vee \dots \vee X_t^{(n)}\right\}_{t \in T}, \quad (5.7)$$

with:

$$X_t^{(j)} := \int_{S^{(j)}}^e f_t(s) M_\alpha(ds), \quad 1 \leq j \leq n, \quad \forall t \in T.$$

Note that given a spectral representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$, the co-spectral decomposition is defined modulo μ -null sets and the induced decomposition is invariant w.r.t. the versions of the decomposition. Namely, if there is another co-spectral decomposition w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$, say $S = \bigcup_{1 \leq j \leq n} \tilde{S}^{(j)} \quad \text{mod } \mu$, then from (5.5) and the disjointness of $\{\mathcal{P}_j\}_{1 \leq j \leq n}$, it follows that $\mu(\tilde{S}^{(j)} \cap S^{(j)}) = 0, 1 \leq j \leq n$. This yields the same decomposition (5.6).

Moreover, the decomposition is invariant w.r.t. the choice of spectral representation.

Theorem 5.2. *Suppose $\{\mathcal{P}_j\}_{1 \leq j \leq n}$ are disjoint positive cones in $L_+^0(T, \mathcal{T}, \lambda)$. For any α -Fréchet process $\{X_t\}_{t \in T}$ with measurable spectral representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$, suppose $\{f_t\}_{t \in T}$ has a co-spectral decomposition w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$. Then,*

- (i) *the decomposition (5.7) is unique in distribution.*
- (ii) *the components $\{X_t^{(j)}\}_{t \in T}, 1 \leq j \leq n$ are independent α -Fréchet processes.*

The proof is given in Appendix. In the special case when $n = 1$, Theorem 5.2 yields the following:

Corollary 5.2. *Let $X = \{X_t\}_{t \in T}$ be an α -Fréchet process with two jointly measurable representations $\{f_t^{(i)}(s)\}_{t \in T} \subset L_+^\alpha(S_i, \mu_i)$, $i = 1, 2$. Consider a positive cone $\mathcal{P} \subset L_+^0(T, \mathcal{T}, \lambda)$. If $f^{(1)}(s) \in \mathcal{P}$, for μ_1 -almost all $s \in S_1$, then $f^{(2)}(s) \in \mathcal{P}$, for μ_2 -almost all $s \in S_2$.*

Corollary 5.2 can be used to distinguish between various α -Fréchet processes in terms of their co-spectral functions. For example, any measurable representation of the α -Fréchet *extremal process* in (5.4) should involve simple indicator-type co-spectral functions with one jump down to zero. The next result shows another application of Corollary 5.2.

Corollary 5.3. *Consider the moving maxima α -Fréchet random fields:*

$$\{X_t\}_{t \in \mathbb{R}^d} \stackrel{d}{=} \left\{ \int_{\mathbb{R}^d}^e f(t-s) M_\alpha(ds) \right\}_{t \in \mathbb{R}^d} \quad \text{and} \quad \{Y_t\}_{t \in \mathbb{R}^d} \stackrel{d}{=} \left\{ \int_{\mathbb{R}^d}^e g(t-s) M_\alpha(ds) \right\}_{t \in \mathbb{R}^d},$$

with $d \in \mathbb{N}$, where f and g belong to $L_+^\alpha(\mathbb{R}^d, \lambda)$. Here M_α is an α -Fréchet random sup-measure on \mathbb{R}^d with the Lebesgue control measure. We have $\{X_t\}_{t \in T} \stackrel{d}{=} \{Y_t\}_{t \in T}$, if and only if $g(x) = f(x + \tau)$, almost all $x \in \mathbb{R}^d$, with some fixed $\tau \in \mathbb{R}^d$.

Proof. The ‘if’ part is trivial. To prove the ‘only if’ part, introduce the cone $\mathcal{P}_f = \{cf(\cdot + \tau), c \geq 0, \tau \in \mathbb{R}^d\}$. Corollary 5.2 implies that $g(\cdot) \in \mathcal{P}_f$, and hence $g(x) = cf(x + \tau)$. Since

$$\|X_0\|_\alpha^\alpha = \int_{\mathbb{R}^d} g^\alpha(x) dx = \int_{\mathbb{R}^d} f^\alpha(x) dx,$$

it follows that $c = 1$. This completes the proof. \square

Theorem 5.2 is a general result in the sense that the cones $\{\mathcal{P}_j\}_{1 \leq j \leq n}$ may be associated with various properties of the co-spectral functions $t \mapsto f_t(s)$ of the process X . If $T \equiv \mathbb{R}^d$, $d \geq 1$, for example, one can consider the cones of co-spectral functions that are: *differentiable, continuous, integrable, or β -Hölder continuous*. Every choice of cones leads to different types of classifications for measurable α -Fréchet processes or fields $X = \{X_t\}_{t \in T}$. We conclude this section by giving two important examples of classifications, motivated by existing results in the literature on sum-stable processes.

Remark 5.1. Note that, instead of (5.6), one may want to define $S^{(j)} := \{s : f(s) \in \mathcal{P}_j\}$, $1 \leq j \leq n$. However, for certain cones, the $S^{(j)}$ ’s defined in this way may not be measurable. See Example 5.4.

Example 5.3 (CONSERVATIVE-DISSIPATIVE DECOMPOSITION). Let $X = \{X_t\}_{t \in T}$ be an α -Fréchet process with measurable representation $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S, \mu)$. Consider the following partition of the set $S = C \cup D$ with

$$C := \left\{ s : s \in S, \int_T f_t^\alpha(s) \lambda(dt) = \infty \right\} \quad \text{and} \quad D := \left\{ s : s \in S, \int_T f_t^\alpha(s) \lambda(dt) < \infty \right\}. \quad (5.8)$$

Note that C and $D = S \setminus C$ are both \mathcal{S} -measurable since $f_t(s)$ is jointly measurable. Observe that this partition of S yields the decomposition:

$$\{X_t\}_{t \in T} \stackrel{d}{=} \{X_t^C \vee X_t^D\}_{t \in T}, \quad (5.9)$$

where $X^C = \{X_t^C\}_{t \in T}$ and $X^D = \{X_t^D\}_{t \in T}$ are defined as:

$$X_t^C = \int_C^e f_t dM_\alpha \quad \text{and} \quad X_t^D = \int_D^e f_t dM_\alpha, \quad \forall t \in T. \quad (5.10)$$

Here M_α is an α -Fréchet random sup-measure with control measure μ .

The decomposition in (5.9) corresponds to the general decomposition in (5.7). Indeed, the co-spectral functions of the component X^D belong to the positive cone of *integrable* functions, while those of X^C belong to the cone of non-integrable functions. By Theorem 5.2, the decomposition (5.9) does not depend on the choice of the representation. The components X^C and X^D of X are independent and they are called the *conservative* and *dissipative* parts of X , respectively. The Decomposition (5.9) is referred to as the *conservative-dissipative* decomposition.

Example 5.4 (POSITIVE-NULL DECOMPOSITION). Following Samorodnitsky [31], consider $T = \mathbb{R}$ or \mathbb{Z} . Introduce the class \mathcal{W} of *positive* weight functions $w : T \rightarrow \mathbb{R}_+$:

$$\mathcal{W} := \left\{ w : \int_T w(t) \lambda(dt) = \infty, \quad w(t) \text{ and } w(-t) \text{ are non-decreasing on } T \cap (0, \infty) \right\}. \quad (5.11)$$

Now we consider the cone

$$\mathcal{P}_{\text{pos}} := \left\{ f \in L_+^0(T, \lambda) : \int_T w(t) f_t^\alpha \lambda(dt) = \infty, \text{ for all } w \in \mathcal{W} \right\}$$

and its complement cone $\mathcal{P}_{\text{null}} := \{0\} \cup (L_+^0(T, \lambda) \setminus \mathcal{P}_{\text{pos}})$.

This choice of cones yields the decomposition

$$\{X_t\}_{t \in T} \stackrel{d}{=} \{X_t^{\text{pos}} \vee X_t^{\text{null}}\}_{t \in T}, \quad (5.12)$$

where

$$X_t^{\text{pos}} := \int_P^e f_t(s) M_\alpha(ds) \quad \text{and} \quad X_t^{\text{null}} := \int_N^e f_t(s) M_\alpha(ds), \quad \forall t \in T, \quad (5.13)$$

with P and N , measurable subsets of S , satisfying $\mu(P \cap N) = 0$, $\mu(S \setminus (P \cup N)) = 0$ and

$$f(s) \in \mathcal{P}_{\text{pos}}, \forall s \in P \quad \text{and} \quad f(s) \in \mathcal{P}_{\text{null}}, \forall s \in N. \quad (5.14)$$

The components $X^{\text{pos}} = \{X_t^{\text{pos}}\}_{t \in T}$ and $X^{\text{null}} = \{X_t^{\text{null}}\}_{t \in T}$ in (5.13) are said to be the *positive* and *null* components of the process X , respectively. By Theorem 5.2, Decomposition (5.12) does not depend on the choice of the measurable representation $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S, \mu)$. It is referred to as the *positive-null* decomposition.

Note that, a technical difference between this example and Example 5.3 is that the set $\tilde{P} := \{s : f(s) \in \mathcal{P}_{\text{pos}}\}$ may not be measurable, even when $f_t(s)$ is jointly measurable.

In the following section, we will study the above decompositions in more detail, for the case of stationary max-stable processes.

6 Classification of Stationary α -Fréchet Processes

In this section, we focus on stationary, measurable max-stable processes $X = \{X_t\}_{t \in T}$, where $T = \mathbb{R}$ or $T = \mathbb{Z}$ is equipped with the Lebesgue or the counting measure λ , respectively. In this case, the process X can be associated with a non-singular flow. Therefore, as in the symmetric α -stable case, the ergodic theoretic properties of the flow yield illuminating structural results.

6.1 Non-singular flows associated with max-stable processes

Following Rosiński [27] (see also Appendix A in [23]), we recall some notions from ergodic theory.

Definition 6.1. A family of functions $\phi = \{\phi_t\}_{t \in T}$, $\phi_t : S \rightarrow S$ for all $t \in T$, is a flow on (S, \mathcal{B}, μ) if

$$(i) \quad \phi_{t_1+t_2}(s) = \phi_{t_2}(\phi_{t_1}(s)), \forall t_1, t_2 \in T, s \in S.$$

$$(ii) \quad \phi_0(s) = s, \forall s \in S.$$

A flow ϕ is said to be *measurable* if $\phi_t(s)$ is a measurable map from $T \times S$ to S ; A flow ϕ is said to be *non-singular* if $\mu(\phi_t^{-1}(A)) = 0 \Leftrightarrow \mu(A) = 0, \forall A \in \mathcal{B}, t \in T$.

The next result relates the spectral functions of stationary α -Fréchet processes to flows.

Theorem 6.1. Let $\{X_t\}_{t \in T}$ be a stationary α -Fréchet process. Suppose that X has a measurable representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$, which is minimal, with standardized support. Then, there exist a unique, modulo $\lambda_{I,N}$, non-singular and measurable flow $\{\phi_t\}_{t \in T}$ such that for each $t \in T$,

$$f_t(s) = \left(\frac{d(\lambda_{I,N} \circ \phi_t)}{d\lambda_{I,N}} \right)^{1/\alpha} (s) (f_0 \circ \phi_t)(s), \quad \lambda_{I,N}\text{-a.e.} \quad (6.1)$$

Theorem 6.1 is stronger than Theorem 6.1 in [8], where the measurability is not considered and the flow structure is not explicitly explored. The proof is given in Appendix A.2. For the readers familiar with Rosiński's work [27], this result is similar to Theorem 3.1 therein. In view of this result, we will say that a stationary α -Fréchet measurable process $\{X_t\}_{t \in T}$ is *generated* by the non-singular measurable flow $\{\phi_t\}_{t \in T}$ on (S, μ) if it has a spectral representation $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$, where:

$$f_t = \left(\frac{d(\mu \circ \phi_t)}{d\mu} \right)^{1/\alpha} (f_0 \circ \phi_t), \quad \mu\text{-a.e.}, \quad (6.2)$$

and

$$\text{supp}\{f_0 \circ \phi_t : t \in T\} = S, \quad \mu\text{-a.e.} \quad (6.3)$$

Note that in the representation (6.2) and (6.3), we do not assume $\{f_t\}_{t \in T}$ to be minimal. However, the minimality plays a crucial role in the proof of the existence of flow representations in Theorem 6.1.

Definition 6.2. We say two measurable non-singular flows $\{\phi_t^{(1)}\}_{t \in T}$ and $\{\phi_t^{(2)}\}_{t \in T}$ on $(S_i, \mu_i), i = 1, 2$, are equivalent, written $\{\phi_t^{(1)}\}_{t \in T} \sim^\Phi \{\phi_t^{(2)}\}_{t \in T}$, if there exists a measurable map $\Phi : S_2 \rightarrow S_1$ such that:

(i) There exist $N_i \subset S_i$ with $\mu_i(N_i) = 0, i = 1, 2$ such that Φ is a Borel isomorphism between $S_2 \setminus N_2$ and $S_1 \setminus N_1$.

(ii) μ_1 and $\mu_2 \circ \Phi^{-1}$ are mutually absolutely continuous.

(iii) $\phi_t^{(1)} \circ \Phi = \Phi \circ \phi_t^{(2)} \mu_2\text{-a.e.}$ for each $t \in T$.

The next result shows the connection between different flows generating the same stationary α -Fréchet process $\{X_t\}_{t \in T}$. The proof is given in Appendix A.2.

Proposition 6.1. *Let $\{X_t\}_{t \in T}$ be a measurable stationary α -Fréchet process.*

(i) *Suppose $\{\phi_t^{(1)}\}_{t \in T}$ is a flow on (S_1, μ_1) and $\{X_t\}_{t \in T}$ is generated by $\{\phi_t^{(1)}\}_{t \in T}$ with spectral function $f_0^{(1)} \in L_+^\alpha(S_1, \mu_1)$. If $\{\phi_t^{(2)}\}_{t \in T}$ is another flow on (S_2, μ_2) and it is equivalent to $\{\phi_t^{(1)}\}_{t \in T}$ via Φ , then $\{X_t\}_{t \in T}$ can also be generated by $\{\phi_t^{(2)}\}_{t \in T}$ with the spectral function*

$$f_0^{(2)}(s) = \left(\frac{d(\mu_1 \circ \Phi)}{d\mu_2}(s) \right)^{1/\alpha} \left(f_0^{(1)} \circ \Phi \right)(s). \quad (6.4)$$

Moreover, if $\{f_t^{(1)}\}_{t \in T}$ is minimal, then $\{f_t^{(2)}\}_{t \in T}$ is minimal.

(ii) *If $\{X_t\}_{t \in T}$ has two measurable minimal representations generated by flows $\{\phi_t^{(i)}\}_{t \in T}$ on (S_i, μ_i) for $i = 1, 2$, then $\{\phi_t^{(1)}\}_{t \in T} \sim^\Phi \{\phi_t^{(2)}\}_{t \in T}$ and (6.4) holds, for some Φ satisfying conditions in Definition 6.2.*

Remark 6.1. Not all flow representations are minimal. Proposition 6.1 shows, however, that any two flows corresponding to minimal representations of the same α -Fréchet process are equivalent.

6.2 Decompositions induced by non-singular flows

The decompositions introduced in Examples 5.3 and 5.4 are motivated by corresponding notions from ergodic theory.

Definition 6.3. Consider a measure space (S, μ) and a measurable, non-singular map $\phi : S \rightarrow S$. A measurable set $B \subset S$ is said to be:

- (i) *wandering*: if $\phi^{-n}(B)$, $n = 0, 1, 2, \dots$ are disjoint.
- (ii) *weakly wandering*: if $\phi^{-n_k}(B)$, $n_k \in \mathbb{N}$ are disjoint, for an infinite sequence $0 = n_0 < n_1 < \dots$.

Now we give two decompositions for max-stable processes. Their counterparts for sum-stable processes have been thoroughly studied (see [27] and [31]).

HOPF (CONSERVATIVE-DISSIPATIVE) DECOMPOSITION. The map ϕ is said to be *conservative* if there is no *wandering* measurable set $B \subset S$, with positive measure $\mu(B) > 0$. One can show that for any measurable, non-singular map $\phi : S \rightarrow S$, there exists a partition of S into two disjoint measurable sets $S = C \cup D$, $C \cap D = \emptyset$ such that: (i) C and D are ϕ -invariant; (ii) $\phi : C \rightarrow C$ is conservative and $D = \bigcup_{k \in \mathbb{Z}} \phi^k(B)$, for some wandering set $B \subset S$. This decomposition is unique (mod μ) and is called the *Hopf decomposition* of S with respect to ϕ . If the component C is trivial, i.e. $\mu(C) = 0$, then ϕ is said to be *dissipative*. The restrictions $\phi : C \rightarrow C$ and $\phi : D \rightarrow D$ are the *conservative* and *dissipative* components of the mapping ϕ , respectively.

Now, given a jointly measurable, non-singular flow $(t, s) \mapsto \phi_t(s)$, $t \in T$, $s \in S$, one can consider the Hopf decompositions $S = C_t \cup D_t$ for each ϕ_t , $t \in T \setminus \{0\}$. By the measurability however, it follows that $\mu(C_t \Delta C) = \mu(D_t \Delta D) = 0$, for some $C \cap D = \emptyset$, $S = C \cup D$ (see e.g. [27, 18]). One thus obtains that any measurable non-singular flow $\{\phi_t\}_{t \in T}$ has a Hopf decomposition $S = C \cup D$, where $\phi^C := \{\phi_t|_C\}_{t \in T}$ and $\phi^D := \{\phi_t|_D\}_{t \in T}$ are *conservative* and *dissipative* flows, respectively.

The following result is an immediate consequence from the proofs of Theorem 4.1 and Corollary 4.2 in Rosiński [27].

Theorem 6.2. Let $X = \{X_t\}_{t \in T}$ be a stationary α -Fréchet process with measurable representation $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S, \mu)$ of full support. Then:

(i) X is generated by a conservative flow, if and only if,

$$\int_T f_t^\alpha(s) \lambda(dt) = \infty, \quad \text{for } \mu\text{-almost all } s \in S;$$

(ii) X is generated by a dissipative flow, if and only if,

$$\int_T f_t^\alpha(s) \lambda(dt) < \infty, \quad \text{for } \mu\text{-almost all } s \in S.$$

(iii) If X is generated by a conservative (dissipative) flow in one representation, then so is the case for any other measurable representation of X .

This result justifies the terminology in the *conservative-dissipative* decomposition of Example 5.3. In particular, the sets C and D in (5.9) correspond precisely to the conservative and dissipative parts in the Hopf decomposition of the flow $\{\phi_t\}_{t \in T}$ associated with the process X .

POSITIVE-NULL DECOMPOSITION. Recall the notion of *weakly wandering* set (Definition 6.3). If one replaces ‘wandering’ by ‘weakly wandering’ in the Hopf decomposition, one obtains the so-called *positive-null decomposition* of S . Alternatively, the map ϕ is said to be *positive*, if there exists a finite measure $\nu \sim \mu$, such that ϕ is ν -invariant. In this case, there are no weakly wandering sets B of positive μ -measure (or equivalently, ν -measure). For any non-singular map ϕ , there exists a partition $S = P \cup N$, unique modulo μ , such that P and N are disjoint, measurable and ϕ -invariant. Furthermore, $\phi : P \rightarrow P$ is positive, and $N = \bigcup_{k \geq 0} \phi^{-n_k}(B)$, for some disjoint $\phi^{-n_k}(B)$ ’s, where B is weakly wandering. The set N (P resp.) is called the null-recurrent (positive-recurrent) part of S , w.r.t. the map ϕ (see e.g. Section 1.4 in [1]).

As in the case of the Hopf decomposition, a jointly measurable, non-singular flow $\{\phi_t\}_{t \in T}$ gives rise to a *positive-null decomposition*: $S = P \cup N$, where $\mu(P_t \Delta P) = \mu(N_t \Delta N) = 0$, for all $t \in T \setminus \{0\}$, and where $S = P_t \cup N_t$ is the positive-null decomposition of the map ϕ_t , $t \in T \setminus \{0\}$ (see e.g. [31, 18]).

Theorem 2.1 of Samorodnitsky [31] about symmetric α -stable processes applies *mutatis mutandis* to the max-stable case:

Theorem 6.3. Let $X = \{X_t\}_{t \in T}$ be a stationary α -Fréchet process with measurable representation $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S, \mu)$ of full support. Then:

(i) X is generated by a positive flow, if and only if, for all $w \in \mathcal{W}$,

$$\int_T w(t) f_t^\alpha(s) \lambda(dt) = \infty, \quad \text{for } \mu\text{-almost all } s \in S,$$

where \mathcal{W} is as in (5.11).

(ii) X is generated by a null flow, if and only if, for some $w \in \mathcal{W}$,

$$\int_T w(t) f_t^\alpha(s) \lambda(dt) < \infty, \quad \text{for } \mu\text{-almost all } s \in S.$$

(iii) If X is generated by a positive (null) flow in one representation, then so is the case for any other measurable representation of X .

As in the Hopf decomposition, Theorem 6.3 shows that the components X^{pos} and X^{null} in the decomposition (5.12) are generated by positive- and null-recurrent flows, respectively. This is because the sets P and N in (5.14) yield the positive-null decomposition of a flow $\{\phi_t\}_{t \in T}$ associated with X .

6.3 Structural results, examples and open questions

Here, we collect some structural results and observations on the interplay between the three types of classifications of max-stable processes discussed above. Namely, (i) continuous-discrete (ii) conservative-dissipative and (iii) positive-null.

Theorems 6.2 and 6.3 imply that the *positive* component of a max-stable process is *conservative* and the *dissipative* one is *null-recurrent*. Thus, for a measurable stationary α -Fréchet process $\{X_t\}_{t \in T}$, we have the decomposition:

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{ X_t^{\text{pos}} \vee X_t^{C, \text{null}} \vee X_t^D \right\}_{t \in T}, \quad (6.5)$$

where $X_t^C = X_t^{\text{pos}} \vee X_t^{C, \text{null}}$ and $X_t^{\text{null}} = X_t^{C, \text{null}} \vee X_t^D$, $t \in T$. Here X^{pos} , $X^{C, \text{null}}$ and X^D are independent α -Fréchet processes. X^{pos} is positive-recurrent and conservative, X^D is dissipative and null-recurrent, and $X^{C, \text{null}}$ is conservative and null-recurrent. We will see that the X^D is precisely the mixed moving maxima. Moreover, we show that the spectrally discrete component has no conservative-null component $X^{C, \text{null}}$.

The following theorem shows that the *purely dissipative* stationary α -Fréchet processes are precisely the *mixed moving maxima*.

Theorem 6.4. *Let $\{X_t\}_{t \in T}$ be a measurable stationary α -Fréchet process. This process is generated by a dissipative flow if and only if there exist a Borel space W , a σ -finite measure ν on W and a function $g \in L_+^\alpha(W \times T, \nu \otimes \lambda)$ such that*

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{ \int_{W \times T}^e g(x, t+u) M_\alpha(dx, du) \right\}_{t \in T}.$$

Here M_α is an α -Fréchet random sup-measure on $W \times T$ with the control measure $\nu \otimes \lambda$ and λ is the Lebesgue measure if $T = \mathbb{R}$ and the counting measure if $T = \mathbb{Z}$. Moreover, one can always choose (W, ν) and g such that the representation $g_t(x, u) := g(x, t+u)$ is minimal.

Proof. Since $g \in L_+^\alpha(W \times T, \nu \otimes \lambda)$, the Fubini's theorem implies $\int_T g(x, t+u)^\alpha \lambda(dt) < \infty$, for almost all $(x, u) \in W \times T$. This, in view of (5.8) implies that X is dissipative..

The ‘only if’ part follows as in the proof of Theorem 4.4 in Rosiński [27] from the results of Krengel [17]. \square

Remark 6.2. Theorem 6.4 parallels the fact that the class of stationary and dissipative symmetric α -stable processes is precisely the class of mixed moving averages (see Theorem 4.4 in [27]). Recently, Kabluchko [15] established the same result as in Theorem 6.4 by using an interesting *association device* between α -Fréchet ($\alpha \in (0, 2)$) and symmetric α -stable processes.

As shown in [34], the mixed moving maxima processes are mixing and hence ergodic. Thus, Theorem 6.4 implies that the dissipative component of a max-stable process is mixing. On the other hand, Samorodnitsky [31] has shown (Theorem 3.1 therein) that

stationary symmetric α -stable processes are ergodic *if and only if* they are generated by a null-recurrent flow. Kabluchko [15] (Theorem 8 therein) has shown that this continues to be the case for stationary α -Fréchet processes.

The previous discussion shows that the ergodic and mixing properties of the null and dissipative components are in line with the decomposition $X_t^{\text{null}} = X_t^D \vee X_t^{C,\text{null}}$, $t \in T$. An example of conservative-null flow can be found in [31]. This yields non-trivial examples of sum- and max-stable processes that are conservative and null. We are not aware, however, of an example of an ergodic max-stable process that is not mixing.

The next two results clarify the structure of the stationary *spectrally discrete* processes in discrete ($T = \mathbb{Z}$) and continuous ($T = \mathbb{R}$) time, respectively. We first show that for *spectrally discrete* stationary max-stable time series, the conservative-dissipative and positive-null decompositions coincide. That is, such processes have no *conservative-null* components. Moreover, the *dissipative* (equivalently *null-recurrent*) component does not exist if the time series has only *finite* number of principal components.

Proposition 6.2. *Let $X = \{X_t\}_{t \in T}$, with $T = \mathbb{Z}$ be a stationary α -Fréchet process (time series).*

- (i) X^N has no conservative-null component, i.e. $X^{N,C,\text{null}} = 0$.
- (ii) If $1 \leq N < \infty$, then X^N is necessarily conservative, and equivalently, positive recurrent.

Proof. Without loss of generality, suppose $X = X^N$ and let $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S_N, \lambda_N)$ be a minimal representation with standardized support for X . We have that

$$f_t = \left(\frac{d(\lambda_N \circ \phi_t)}{d\lambda_N} \right)^{1/\alpha} f_0 \circ \phi_t,$$

where $\phi_t : S_N \rightarrow S_N$ is a non-singular flow on (S_N, λ_N) . Since $S_N \subset \mathbb{N}$ and λ_N is the counting measure, the non-singular transformations are necessarily measure-preserving, i.e., permutations. Thus the term $d(\lambda_N \circ \phi_t)/d\lambda_N \equiv 1$ and $f_t(s) = f_0 \circ \phi_t(s)$.

We start by proving (ii). Since $\phi_1 : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ is a permutation, it has a finite invariant measure and hence the flow $\{\phi_t\}_{t \in T}$ is positive-recurrent and hence conservative.

Now we prove (i). Note that when $1 \leq N < \infty$, we have shown in (ii) that X^N is conservative and positive-recurrent. For $N = \infty$, we consider two cases. First we suppose that for every $s \in S_N$, the recurrent time

$$\tau_s := \inf\{t > 0 : \phi_t(s) = s\} \tag{6.6}$$

is finite. Let $\mathfrak{D}(s)$ denote the orbit of state s w.r.t. flow $\{\phi_t\}_{t \in T}$, i.e., $\mathfrak{D}(s) := \{\phi_t(s) : t \in T\}$. Every orbit of $\{\phi_t\}_{t \in T}$ is τ_s -periodic, i.e., $|\mathfrak{D}(s)| < \infty$. Since $N = \infty$, the total number of different orbits must be infinite. Enumerate all the orbits by $\mathfrak{D}_1, \mathfrak{D}_2, \dots$, so that $\mathfrak{D}(s) = \mathfrak{D}_{\pi(s)}$ with $\pi : S_N \rightarrow \mathbb{N}$ and $S_N = \bigcup_{k \in \mathbb{N}} \mathfrak{D}_k$. Observe that the orbits are disjoint. We now define a finite invariant measure on S_N , equivalent to the counting measure:

$$\tilde{\lambda}(\{s\}) := 2^{-\pi(s)} \frac{1}{|\mathfrak{D}_{\pi(s)}|}, \forall s \in S_N.$$

This measure is clearly invariant on each \mathfrak{D}_k , for all $k \in \mathbb{N}$. Since $\tilde{\lambda}(\mathfrak{D}_k) = 2^{-k}$, the measure $\tilde{\lambda}$ is finite and it is clearly equivalent to the counting measure. Thus, X^N is positive and conservative.

On the other hand, suppose that there exists a state s with $\tau_s = \infty$. Then, its orbit is infinite and non-recurrent, i.e., $|\mathfrak{D}_k(s)| = \infty$. Then, the flow $\{\phi_t\}_{t \in T}$ is both null-recurrent and dissipative on $\mathfrak{D}_k(s)$. Indeed, the null recurrence follows from the fact that there is no positive finite invariant measure on $\mathfrak{D}_k(s)$. The dissipativity follows from the remark that $\mathfrak{D}_k(s) = \bigcup_{j \in \mathbb{Z}} \phi_j(s)$ is a disjoint union. We have thus shown that $\{\phi_t\}_{t \in T}$ is dissipative and null-recurrent on non-recurrent orbits. \square

The following result shows that the *continuous-time* stationary, measurable and *spectrally discrete* max-stable processes are trivial.

Theorem 6.5. *Let $X = \{X_t\}_{t \in T}$, with $T = \mathbb{R}$ be a stationary and measurable α -Fréchet process. If $N \geq 1$, then it must be $N = 1$. That is, the spectrally discrete component X^N is the random constant process: $\{X_t^N\}_{t \in \mathbb{R}} \stackrel{d}{=} \{Z\}_{t \in \mathbb{R}}$, for some α -Fréchet variable Z .*

Proof. Let $\{f_t\}_{t \in T}$ and $\{\phi_t\}_{t \in T}$ be as in Proposition 6.2. Observe moreover that, in this case, the ϕ_t 's are measure-preserving bijections, and in view of Theorem 6.1, the flow $\{\phi_t(s)\}$ is measurable. For any fixed $s \in S_N$, consider τ_s defined in (6.6). The proof consists of three steps.

(i) *We show first that $\tau_s = 0$ implies $\phi_t(s) \equiv s$, for all $t \in \mathbb{R}$.* Indeed, suppose that $\tau_s = 0$ and note that, by definition, for all $n > 0$, there exist $0 < t_{n,1} < t_{n,2} < 1/n$ such that $\phi_{t_{n,1}}(s) = \phi_{t_{n,2}}(s) = s$. Set $T_0 := \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} \{t_{n,1} + k(t_{n,2} - t_{n,1})\}$. It follows that T_0 is dense in \mathbb{R} and $\phi_t(s) = s$, for all $t \in T_0$. Hence $f_t(s) = f_0 \circ \phi_t(s) = f_0(s)$, for all $t \in T_0$. Now, we define a new α -Fréchet process $Y = \{Y_t\}_{t \in T}$:

$$\{Y_t\}_{t \in T} \stackrel{d}{=} \left\{ \int_{S_N}^e \mathbf{1}_{\{\cdot=s\}} \circ \phi_t(r) M_\alpha(dr) \right\}_{t \in T}.$$

Since $\{\phi_t\}_{t \in T}$ is a flow, ϕ_t is invertible, for any $t \in T_0$. Hence, for all $t \in T_0$, we have $\phi_t(r) = \phi_t(s) \equiv s$ if and only if $r = s$. This shows that, for all $t \in T_0$,

$$\mathbf{1}_{\{\cdot=s\}} \circ \phi_t(r) \equiv \mathbf{1}_{\{\phi_t(r)=s\}} = \mathbf{1}_{\{r=s\}} \equiv \mathbf{1}_{\{\cdot=s\}} \circ \phi_0(r),$$

which implies that $Y_t = Y_0$, almost surely, $\forall t \in T_0$. Moreover, as $\{\phi_t\}_{t \in T}$ is measurable, so is $\{Y_t\}_{t \in T}$ by Proposition 5.2. Also, $Y = \{Y_t\}_{t \in T}$ is stationary, since it is generated by a measure preserving flow. Thus, the stationarity and measurability of Y imply that it is *continuous in probability* (see Theorem 3.1 in [34]). This, and the fact that $Y_t = Y_0$, a.s., for all t in a dense sub-set T_0 of \mathbb{R} , imply that $Y_t = Y_0$, a.s., for all $t \in \mathbb{R}$. Therefore, for the spectral functions, we obtain $\mathbf{1}_{\{\phi_t(r)=s\}} = \mathbf{1}_{\{r=s\}}$, $\forall r \in S_N, t \in \mathbb{R}$. This shows that $\phi_t(s) = s$, $\forall t \in \mathbb{R}$.

(ii) *We show next that $\tau_s > 0$ implies $\phi_{\tau_s}(s) = s$.* Suppose that $\phi_{\tau_s}(s) \neq s$. Then, as above, there exist $t_1, t_2 \in (\tau_s, \tau_s + \tau_s/2)$ such that $\phi_{t_1}(s) = \phi_{t_2}(s) = s$. But it follows that $\phi_{t_1+k(t_2-t_1)}(s) = s$ for all $k \in \mathbb{Z}$. This, since $\{t_1 + k(t_2 - t_1)\}_{k \in \mathbb{Z}} \cap (0, \tau_s) \neq \emptyset$, contradicts the definition of τ_s .

(iii) *Now, we show that it is impossible to have $\tau_s > 0$ for all $s \in S_N$.* Write $\mathfrak{T}_s = \{t : \phi_t(s_0) = s, \text{ for some } s_0 \in S_N\}$. Observe that the set \mathfrak{T}_s is countably infinite for all $s \in S_N$ such that $\tau_s > 0$, since by (ii) above, $\mathfrak{T}_s = \{k\tau_s\}_{k \in \mathbb{Z}}$. Note also that $\bigcup_{s \in S_N} \mathfrak{T}_s = \mathbb{R}$. However, the assumption that $\tau_s > 0$ for all $s \in S_N$ would imply $\bigcup_{s \in S_N} \mathfrak{T}_s$ has cardinality of \mathbb{N} equals that of \mathbb{R} , which is a contradiction.

We now conclude the proof. By (iii) above, there must exist $s \in S_N$ such that $\tau_s = 0$. Set $\mathfrak{R} = \{s \in S_N : \phi_t(s) = s, \forall t \in \mathbb{R}\}$. We have already seen in (i) that $\tau_s = 0$ implies $\phi_t(s) \equiv s$, for all $t \in \mathbb{R}$, whence \mathfrak{R} is ϕ -invariant. Consider now a new α -Fréchet process

$$\{Y_t\}_{t \in T} \stackrel{d}{=} \left\{ \int_{S_N \setminus \mathfrak{R}}^e f_t(r) M_\alpha(dr) \right\}_{t \in T}.$$

Since the f_t 's, restricted to the ϕ -invariant set $S_N \setminus \mathfrak{R}$ yield a minimal representation for $Y = \{Y_t\}_{t \in T}$ with standardized support. This process is generated by the same flow $\{\phi_t\}_{t \in \mathbb{R}}$, restricted to $S_N \setminus \mathfrak{R}$. Since $\tau_s > 0$, $\forall s \in S_N \setminus \mathfrak{R}$, by (ii), it follows that $S_N \setminus \mathfrak{R} = \emptyset$.

On the other hand, since $\phi_t(s) \equiv s, \forall t \in \mathbb{R}, s \in \mathfrak{R}$, the minimality of $\{f_t\}_{t \in T}$ implies that $|\mathfrak{R}| = |S_N| = 1$. Therefore, $\{X_t\}_{t \in T} \stackrel{d}{=} \{Z\}_{t \in T}$ for some α -Fréchet random variable Z . \square

Example 6.1. In contrast with Proposition 6.2 (i), the spectrally discrete component of a stationary α -Fréchet time series may be dissipative if it involves infinite number of principal components. Indeed, by Theorem 6.4, the moving maxima $X_t := \int_{\mathbb{Z}}^e f(t+s) M_\alpha(ds) \equiv \bigvee_{i \in \mathbb{Z}} f(t+i) M_\alpha(\{i\})$, is dissipative and spectrally discrete, where M_α has the counting control measure on \mathbb{Z} .

Example 6.2. Suppose that (E, \mathcal{E}, μ) is a probability space, i.e. $\mu(E) = 1$. Let M_α be an α -Fréchet random sup-measure on E with control measure μ , which is defined on a *different* probability space. Suppose that $\{Y_t\}_{t \in T}$ is a positive stochastic process on (E, \mathcal{E}, μ) such that $\mathbb{E}_\mu Y_t^\alpha < \infty$, for all $t \in T$. Then, the α -Fréchet process:

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{ \int_E^e Y_t(u) M_\alpha(du) \right\}_{t \in T}, \quad (6.7)$$

is said to be *doubly stochastic*.

One can show that, in (6.7), if $\{Y_t\}_{t \in T}$ is stationary, then so is $\{X_t\}_{t \in T}$. The Brown-Resnick processes discussed in next section shows that the converse is not always true.

7 Brown-Resnick Processes

Consider the following *doubly stochastic process* (see e.g. [16] and [34]):

$$\{X_t\}_{t \in \mathbb{R}} \stackrel{d}{=} \left\{ \int_E^e e^{W_t - \sigma_t^2/2} dM_1 \right\}_{t \in \mathbb{R}}. \quad (7.1)$$

Here W_t is a zero-mean Gaussian process defined on the probability space (E, \mathcal{E}, μ) with variance σ_t^2 . Since $\mathbb{E}_\mu \left(e^{W_t - \sigma_t^2/2} \right) = 1 < \infty$, the 1-Fréchet process in (7.1) is well-defined. The processes having representation (7.1) were first introduced by Brown and Resnick [3] with W_t being the standard Brownian motion. In general, we will call $\{X_t\}_{t \in \mathbb{R}}$ as in (7.1) a *Brown-Resnick 1-Fréchet process*.

Kabluchko *et al.* [16] have shown that if $\{W_t\}_{t \in \mathbb{R}}$ has *stationary increments*, then the Brown-Resnick process $\{X_t\}_{t \in \mathbb{R}}$ in (7.1) is stationary. The following interesting result about an arbitrary zero-mean Gaussian process with stationary increments and continuous paths is obtained by combining the results of [16] and our Theorems 6.2 and 6.4 above.

Theorem 7.1. *Let $W = \{W_t\}_{t \in \mathbb{R}}$ be a Gaussian zero-mean process with stationary increments and continuous paths. If*

$$\lim_{|t| \rightarrow \infty} (W_t - \sigma_t^2/2) = -\infty, \quad \text{almost surely} \quad (7.2)$$

then,

$$\int_{-\infty}^{\infty} e^{W_t - \sigma_t^2/2} dt < \infty, \quad \text{almost surely}, \quad (7.3)$$

where $\sigma_t^2 = \mathbb{E}W_t^2 = \text{Var}(W_t)$.

Proof. Let $\{X_t\}_{t \in \mathbb{R}}$ be the Brown–Resnick process defined in (7.1). Note that the process $\{\log X_t\}_{t \in \mathbb{R}}$ is also max-stable but it has Gumbel marginals. Kabluchko *et al.* [16] have shown that $\{\log X_t\}_{t \in \mathbb{R}}$ is stationary and hence so is $\{X_t\}_{t \in \mathbb{R}}$. Moreover, by Theorem 13 in [16], Condition (7.2) implies that $\{\log X_t\}_{t \in \mathbb{R}}$, or equivalently, $\{X_t\}_{t \in \mathbb{R}}$, has a mixed moving maxima representation. On the other hand, Theorem 6.4 implies that any process with mixed moving maxima representation is dissipative. Dissipativity of $\{X_t\}_{t \in \mathbb{R}}$ is equivalent to (7.3) by Theorem 6.2. This completes the proof. \square

The following question arises.

Question 7.1. *For what general classes of continuous-path, zero mean Gaussian processes $\{W_t\}_{t \in \mathbb{R}}$ with stationary increments, is the Brown–Resnick stationary process (7.1) purely dissipative?*

The next result provides a *partial* answer to this question for the interesting case when $W = \{W_t\}_{t \in \mathbb{R}}$ is the fractional Brownian motion (fBm). Recall that the fBm is a zero-mean Gaussian processes with stationary increments, which is self-similar. The process W is said to be self-similar with self-similarity parameter $H > 0$, if for all $c > 0$, we have that $\{W_{ct}\}_{t \in \mathbb{R}} \stackrel{d}{=} \{c^H W_t\}_{t \in \mathbb{R}}$. The fBm necessarily has the covariance function

$$\mathbb{E}W_t W_s = \frac{\sigma^2}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad \text{with } t, s \in \mathbb{R}, \quad (7.4)$$

where $0 < H \leq 1$ is the self-similarity parameter of W . The fractional Brownian motions have versions with continuous paths (see e.g. [32]).

Proposition 7.1. *The stationary Brown–Resnick processes $X = \{X_t\}_{t \in \mathbb{R}}$ associated with the fractional Brownian motions $\{W_t\}_{t \in \mathbb{R}}$ in (7.4) are purely dissipative and hence they have mixed moving maxima representations.*

Proof. Without loss of generality, we will suppose that the fBm W has continuous paths. As indicated above, the stationarity of X follows from the fact that W has stationary increments (see Kabluchko *et al.* [16]). Now, by Theorem 6.2, X is dissipative, if and only if

$$\int_{-\infty}^{\infty} \exp \{W_t - \sigma_t^2/2\} dt < \infty, \quad \text{almost surely}. \quad (7.5)$$

It is enough to focus on the integral $\int_0^{\infty} \exp \{W_t - \sigma_t^2/2\} dt$. By the Law of the Iterated Logarithm for fractional Brownian motion (see Oodaira [21]), we have

$$\limsup_{t \rightarrow \infty} W_t / \sqrt{2\sigma_t^2 \log \log t} = 1, \quad \text{almost surely}.$$

Hence, with probability one, for any $\delta > 0$, there exists T_1 (possibly random) such that $\forall t > T_1$, we have $W_t < (1 + \delta)\sqrt{2\sigma_t^2 \log \log t}$ almost surely. Moreover, there exists T_2 sufficiently large (possibly random), such that $\forall t > T_2$, we have

$$(1 + \delta)\sqrt{2\sigma_t^2 \log \log t} < \sigma_t^2/4 \equiv \sigma^2 t^{2H}/4 \text{ almost surely,}$$

where $H \in (0, 1]$ is the self-similarity parameter of the fractional Brownian motion W . Now, let $T_0 = \max(T_1, T_2)$. It follows that

$$\int_{T_0}^{\infty} e^{W_t - \sigma_t^2/2} dt < \int_{T_0}^{\infty} e^{(1+\delta)\sqrt{2\sigma_t^2 \log \log t} - \sigma^2 t^{2H}/2} dt \leq \int_{T_0}^{\infty} e^{-\sigma^2 t^{2H}/4} dt < \infty \text{ almost surely,}$$

which implies (7.5) since W_t is continuous, with probability one.. \square

Observe that the above result continues to hold even in the degenerate case $H = 1$. One then has that $W_t = tZ$, $t \in \mathbb{R}$, where Z is a zero-mean Gaussian random variable. In this case, the corresponding Brown–Resnick process has a simple *moving maxima* representation. Indeed, for simplicity, let $\sigma^2 = \text{Var}(Z) = 1$ and observe that

$$X_t := \int_E e^{tZ(u) - t^2/2} M_1(du) = \int_E e^{Z^2(u)/2} e^{-(t-Z(u))^2/2} M_1(du).$$

Note that the measure $\nu(A) := \int_E \mathbf{1}_{\{Z(u) \in A\}} e^{Z^2(u)/2} \mu(du) \equiv \lambda(A)/\sqrt{2\pi}$, is up to a constant factor equal to the Lebesgue measure λ on \mathbb{R} . Therefore, one can show that

$$\{X_t\}_{t \in \mathbb{R}} \stackrel{d}{=} \left\{ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(t-z)^2/2} \widetilde{M}_1(dz) \right\}_{t \in \mathbb{R}},$$

where \widetilde{M}_1 is a 1–Fréchet random sup-measure with the Lebesgue control measure. This shows that X in this simple case is merely a *moving maxima* rather than a *mixed moving maxima*.

We have thus shown that the Brown–Resnick process (7.1) driven by fractional Brownian motion $\{W_t\}_{t \in T}$ is purely dissipative. Thus, by Theorem 6.4 we have that $\{X_t\}_{t \in T}$ is a *mixed moving maxima*. It is not clear how one can prove this fact without the use of our classification results. In two very recent papers [16, 15], Kabluchko and co-authors established very similar classification results by using very different methods based on Poisson point processes on abstract path-spaces. Their approach yields directly the moving-maxima representation (and hence dissipativity) of the Brown–Resnick type processes X under the alternative Condition (7.2). This condition is only shown to be *sufficient* for dissipativity of X . Its relationship with our *necessary and sufficient condition* (7.3) is a question of independent interest.

The question raised in Kabluchko [15] on whether there exist stationary Brown–Resnick processes X of mixed type i.e. with non-trivial dissipative and conservative components still remains open. In view of our new necessary and sufficient condition (7.3), this question is *equivalent* to the following:

Question 7.2. *Is it true for Gaussian processes $W = \{W_t\}_{t \in \mathbb{R}}$ with stationary increments and continuous paths that $\mu\{\int_{-\infty}^{\infty} e^{W_t - \sigma_t^2/2} dt < \infty\} \in \{0, 1\}$?*

A Proofs and Auxiliary Results

A.1 Proofs and auxiliary results for Section 3

The proof of Theorem 3.2, as well as the auxiliary results in Section 3, follow closely the proofs in Hardin [12]. There the author dealt with linear isometries instead of max-linear isometries.

Lemma A.1. *Let $(S_1, \mathcal{S}_1, \mu_1)$ and $(S_2, \mathcal{S}_2, \mu_2)$ be two measure spaces and suppose that $T : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is a regular set isomorphism.*

(i) *The regular set isomorphism T induces a canonical function map Tf , defined mod μ_2 for all measurable f on (S_1, \mathcal{S}_1) and such that $\{Tf \in B\} = T\{f \in B\}$, mod μ_2 , for all Borel sets $B \in \mathcal{B}_{\mathbb{R}}$.*

The function map T is monotone, linear, max-linear, and preserves the convergence almost everywhere, modulo null sets. Moreover, $T(f_1 f_2) = T(f_1)T(f_2)$, mod μ_2 , for all measurable functions f_1 and f_2 on (S_1, \mathcal{S}_1) .

(ii) *If the regular set isomorphism T is measure preserving, then the induced function map T is a max-linear isometry from $L_+^\alpha(S_1, \mathcal{S}_1, \mu_1)$ to $L_+^\alpha(S_2, \mathcal{S}_2, \mu_2)$, for every $\alpha > 0$. Furthermore, if T is onto, then so is the induced max-linear isometry.*

Proof. (i) Consider the sets $A_r := \{f \leq r\}$, $r \in \mathbb{Q}$, where \mathbb{Q} denotes the set of rational numbers. By the monotonicity of the regular set isomorphism, we obtain that $TA_r \subset TA_s$, mod μ_2 , for all $r < s$, $r, s \in \mathbb{Q}$. Let $(Tf)(x) := \inf\{s : x \in TA_s\}$ and observe that Tf is measurable. Indeed,

$$\{Tf \leq r\} = \bigcap_{s > r, s \in \mathbb{Q}} TA_s = TA_r = T\{f \leq r\}, \quad \text{for all } r \in \mathbb{Q}.$$

We will show next that $T\{f \in B\} = \{Tf \in B\}$ mod μ_2 , for all Borel sets $B \subset \mathbb{R}$. Indeed, consider the class of sets:

$$\mathcal{D} := \{B \in \mathcal{B}_{\mathbb{R}} : T\{f \in B\} = \{Tf \in B\}, \text{ mod } \mu_2\},$$

and observe that $\mathcal{C} \subset \mathcal{D}$, where $\mathcal{C} := \{(-\infty, r] : r \in \mathbb{Q}\}$. One can show that \mathcal{D} is a σ -algebra and therefore $\sigma(\mathcal{C}) \subset \mathcal{D}$. This however implies that $\mathcal{B}_{\mathbb{R}} \equiv \mathcal{D}$, since $\mathcal{B}_{\mathbb{R}} \equiv \sigma(\mathcal{C})$, thereby showing that $T\{f \in B\} = \{Tf \in B\}$, for all $B \in \mathcal{B}_{\mathbb{R}}$.

By the properties of regular set isomorphism, it is easy to see that for any sequence of measurable functions $\{f_m\}_{m \in \mathbb{N}}$, $T(\sup_{m \in \mathbb{N}} f_m) = \sup_{m \in \mathbb{N}} Tf_m$, and $T(\inf_{m \in \mathbb{N}} f_m) = \inf_{m \in \mathbb{N}} Tf_m$, mod μ_2 . Therefore, T preserves pointwise limits of measurable functions, modulo null sets.

One clearly has that $T(\lambda f) = \lambda Tf$, for all $\lambda \in \mathbb{R}$ and measurable f 's. The linearity of T follows then from the fact that

$$T\{f + g \leq r\} = T\left(\bigcup_{s \in \mathbb{Q}} \{f \leq r - s\} \cap \{g \leq s\}\right) = \bigcup_{s \in \mathbb{Q}} \{Tf \leq r - s\} \cap \{Tg \leq s\},$$

which equals $\{Tf + Tg \leq r\}$. The max-linearity of T can be established similarly. The fact that T preserves products, i.e. $T(f_1 f_2) = T(f_1)T(f_2)$ mod μ_2 , for measurable f_1 and f_2 can be established similarly for non-negative functions, and then shown to hold for arbitrary functions, by linearity.

(ii) Now, if T is measure preserving, then for all simple functions $f = \bigvee_{i=1}^n \lambda_i \mathbf{1}_{A_i}$, with

disjoint A_i 's in \mathcal{S}_1 , we have, by max linearity, that $Tf = \bigvee_{i=1}^n \lambda_i \mathbf{1}_{TA_i}$. Thus, since the regular set isomorphism T is measure preserving, we have

$$\left\| T\left(\bigvee \lambda_i \mathbf{1}_{A_i}\right) \right\|_{L_+^\alpha(\mu_2)} = \left\| \bigvee \lambda_i \mathbf{1}_{T(A_i)} \right\|_{L_+^\alpha(\mu_2)} = \left\| \bigvee \lambda_i \mathbf{1}_{A_i} \right\|_{L_+^\alpha(\mu_1)}.$$

Now, for any $f \in L_+^\alpha(S_1, \mathcal{S}_1, \mu_1)$, let $\{f_t\}_{t \in T} \subset L_+^\alpha(S_1, \mathcal{S}_1, \mu_1)$ be a monotone sequence of simple functions such that $f_n \nearrow f \bmod \mu_1$ as $n \rightarrow \infty$. We then have that $Tf_n \nearrow Tf$, $\bmod \mu_2$, as $n \rightarrow \infty$, and $\int_{S_2} (Tf_n)^\alpha d\mu_2 = \int_{S_1} f_n^\alpha d\mu_1 \nearrow \int_{S_1} f^\alpha d\mu_1 < \infty$, as $n \rightarrow \infty$. The monotone convergence theorem implies that $Tf \in L_+^\alpha(S_2, \mathcal{S}_2, \mu_2)$ and $\|Tf\|_{L_+^\alpha(\mu_2)} = \|f\|_{L_+^\alpha(\mu_1)}$, which shows that T is a max-linear isometry. If T is *onto* as a regular set isomorphism, then the induced max-linear isometry is clearly *onto* the set of all simple functions, and therefore T is *onto* $L_+^\alpha(S_2, \mathcal{S}_2, \mu_2)$. \square

The following lemma is used repeatedly in the sequel.

Lemma A.2. *Let \mathcal{F} be a max-linear sub-space of $L_+^\alpha(S_1, \mu_1)$, where μ_1 is a finite measure. Let $U : \mathcal{F} \rightarrow L_+^\alpha(S_2, \mu_2)$ be a max-linear isometry. If $\mathbf{1}_{S_1} \in \mathcal{F}$ and $U\mathbf{1}_{S_1} = \mathbf{1}_{S_2}$, then for any collection of functions $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$,*

$$\mu_1(\{f_1, f_2, \dots\} \in B) = \mu_2(\{Uf_1, Uf_2, \dots\} \in B), \quad \forall B \in \mathcal{B}_{\mathbb{R}_+^\mathbb{N}}. \quad (\text{A.1})$$

Here $\mathcal{B}_{\mathbb{R}_+^\mathbb{N}}$ denotes the Borel σ -algebra on the product space $\mathbb{R}_+^\mathbb{N} = [0, \infty)^\mathbb{N}$.

Proof. Since $\mathbf{1}_{S_2} \in L_+^\alpha(S_2, \mu_2)$, it follows that μ_2 is a finite measure. Without loss of generality, suppose μ_1 and μ_2 are probability measures. Let $\mu_{1,\infty}(B) = \mu_1(\{f_1, f_2, \dots\} \in B)$ and $\mu_{2,\infty}(B) = \mu_2(\{Uf_1, Uf_2, \dots\} \in B)$, $\forall B \in \mathcal{B}_{\mathbb{R}_+^\mathbb{N}}$ and $f_i \in \mathcal{F}, i \in \mathbb{N}$. In order to show $\mu_{1,\infty} = \mu_{2,\infty}$, we first show that $\mu_{1,\infty}$ and $\mu_{2,\infty}$ induce the same measure $\mu_{1,n}$ and $\mu_{2,n}$ on \mathbb{R}_+^n via $\mu_{i,n}(B_n) = \mu_{i,\infty}(\overline{B}_n)$ for $i = 1, 2$, where $B_n \in \mathcal{B}_{\mathbb{R}_+^n}$ and $\overline{B}_n = \{x = (x_1, x_2, \dots) \in \mathbb{R}_+^\mathbb{N} : (x_1, \dots, x_n) \in B_n\}$. Indeed, by using a change of variables and the fact that U is a max-linear isometry, we obtain that for all $n \in \mathbb{N}$, $a_i > 0, f_i \in \mathcal{F}, 1 \leq i \leq n$,

$$\begin{aligned} \int_{\mathbb{R}_+^\mathbb{N}} \mathbf{1} \vee \left(\bigvee_{1 \leq i \leq n} a_i z_i \right)^\alpha d\mu_{1,\infty}(z) &= \int_{S_1} \mathbf{1} \vee \left(\bigvee_{1 \leq i \leq n} a_i f_i(x) \right)^\alpha d\mu_1(x) \\ &= \int_{S_2} \mathbf{1} \vee \left(\bigvee_{1 \leq i \leq n} a_i Uf_i(y) \right)^\alpha d\mu_2(y) = \int_{\mathbb{R}_+^\mathbb{N}} \mathbf{1} \vee \left(\bigvee_{1 \leq i \leq n} a_i z_i \right)^\alpha d\mu_{2,\infty}(z), \end{aligned}$$

which implies $\mu_{1,n}(B_n) = \mu_{2,n}(B_n), \forall B_n \in \mathcal{B}_{\mathbb{R}_+^n}$. By Lemma 4.1 in [8], $\mu_{1,n} = \mu_{2,n}, \forall n \in \mathbb{N}$. That is, $\mu_{1,\infty}$ and $\mu_{2,\infty}$ agree on the field of all cylinder sets of $\mathbb{R}_+^\mathbb{N}$. By the Carathéodory extension theorem, $\mu_{1,\infty} = \mu_{2,\infty}$. \square

Proof of Theorem 3.1. First observe that μ_1 and μ_2 are finite measures, since $\mathbf{1}_{S_1} \in L_+^\alpha(S_1, \mu_1)$ and $\mathbf{1}_{S_2} \in L_+^\alpha(S_2, \mu_2)$. We start by defining T on a separable σ -field and then we verify the consistency of the definition. Let $C = \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ be a countable collection of functions. We then have that

$$\sigma(C) = \left\{ \{(f_1, f_2, \dots) \in B\}, B \in \mathcal{B}_{\mathbb{R}_+^\mathbb{N}} \right\}, \quad (\text{A.2})$$

where $\sigma(C)$ is the minimal σ -algebra generated by C . Observe that for any $A \in \sigma(C)$, there exists $B_A \in \mathcal{B}_{\mathbb{R}_+^{\mathbb{N}}}$ such that $A = \{(f_1, f_2, \dots) \in B_A\}$. We therefore define $T_C : \sigma(C) \rightarrow \sigma(U(C))$ as

$$T_C A = \{(Uf_1, Uf_2, \dots) \in B_A\}. \quad (\text{A.3})$$

In the sequel, we will first show that T_C is: (1) well defined (note that B_A may not be unique for a given A), (2) measure preserving, (3) onto and (4) T_C induces a unique max-linear isometry from $L_+^\alpha(S_1, \sigma(C), \mu_1)$ onto $L_+^\alpha(S_2, \sigma(U(C)), \mu_2)$, implying that T_C satisfies (i) and (ii). Then we show that for the induced max-linear isometry T_C : (5) T_C and U coincides on $\overline{\text{V-span}}(f_t, t \in T)$ and (6) T_C is unique, implying that T_C satisfies (iii).

(1) T_C is well defined modulo μ_2 -null sets. Indeed, if there is another $\tilde{B}_A \in \mathcal{B}_{\mathbb{R}_+^{\mathbb{N}}}$ such that $A = \{(f_1, f_2, \dots) \in \tilde{B}_A\}$, then

$$\mu_2(\{(Uf_1, Uf_2, \dots) \in B_A\} \Delta \{(Uf_1, Uf_2, \dots) \in \tilde{B}_A\}) = \mu_2(\{(Uf_1, Uf_2, \dots) \in B_A \Delta \tilde{B}_A\}).$$

By Lemma A.2, the last expression equals

$$\begin{aligned} & \mu_1(\{(f_1, f_2, \dots) \in B_A \Delta \tilde{B}_A\}) \\ &= \mu_1(\{(f_1, f_2, \dots) \in B_A\} \Delta \{(f_1, f_2, \dots) \in \tilde{B}_A\}) = \mu_1(A \Delta A) = 0, \end{aligned}$$

which shows that the definition of $T_C A$ in (A.3) does not depend on the choice of $B_A \in \mathcal{B}_{\mathbb{R}_+^{\mathbb{N}}}$, modulo μ_2 -null sets.

(2) T_C is a regular set isomorphism. For convenience, we prove the three conditions in Definition 3.2 in different order. For every $A \in \sigma(C)$, let $B_A \in \mathcal{B}_{\mathbb{R}_+^{\mathbb{N}}}$ be such that $\{(f_1, f_2, \dots) \in B_A\} = A$. First, by Lemma A.2, we have

$$\mu_1(A) = \mu_1(\{(f_1, f_2, \dots) \in B_A\}) = \mu_2(\{(Uf_1, Uf_2, \dots) \in B_A\}) = \mu_2(T_C(A)), \quad (\text{A.4})$$

which shows that (iii) of Definition 3.2 holds. Relation (A.4) shows moreover that T_C is *measure-preserving*. Second, note that $S_1 \in \sigma(C)$. Then we can show (i) of Definition 3.2 by

$$\begin{aligned} T_C(S_1 \setminus A) &= \{(Uf_1, Uf_2, \dots) \in \mathbb{R}_+^{\mathbb{N}} \setminus B_A\} \\ &= \{(Uf_1, Uf_2, \dots) \in \mathbb{R}_+^{\mathbb{N}}\} \setminus \{(Uf_1, Uf_2, \dots) \in B_A\} = T_C(S_1) \setminus T_C(A). \end{aligned}$$

Finally, suppose A_1, A_2, \dots are arbitrary disjoint sets in $\sigma(C)$. Observe that, modulo μ_1 ,

$$\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} \{(f_1, f_2, \dots) \in B_{A_n}\} = \{(f_1, f_2, \dots) \in \cup_{n=1}^{\infty} B_{A_n}\},$$

whence, $\cup_{n=1}^{\infty} B_{A_n}$ may be viewed as a particular choice of $B_{\cup_{n=1}^{\infty} A_n}$, and thus, in view of (A.3),

$$\begin{aligned} T_C(\cup_{n=1}^{\infty} A_n) &= \{(Uf_1, Uf_2, \dots) \in \cup_{n=1}^{\infty} B_{A_n}\} \\ &= \cup_{n=1}^{\infty} \{(Uf_1, Uf_2, \dots) \in B_{A_n}\} = \bigcup_{n=1}^{\infty} \{T_C(A_n)\}, \end{aligned}$$

implying (ii) of Definition 3.2.

(3) T_C is onto $\sigma(U(C))$. Observe first that as in (A.2), $\sigma(U(C)) = \{(Uf_1, Uf_2, \dots) \in \mathcal{B}_{\mathbb{R}_+^{\mathbb{N}}}\}$. Now $\forall U_A \in \sigma(U(C))$ there is a $B_A \in \mathcal{B}_{\mathbb{R}_+^{\mathbb{N}}}$ such that $U_A = \{(Uf_1, Uf_2, \dots) \in B_A\}$. Then, let $A = \{(f_1, f_2, \dots) \in B_A\}$ and observe that by (A.3), we have $T_C A = \{(Uf_1, Uf_2, \dots) \in B_A\} = U_A$. This shows that T_C is onto.

(4) T_C induces a max-linear isometry from $L_+^\alpha(S_1, \sigma(C), \mu_1)$ onto $L_+^\alpha(S_2, \sigma(U(C)), \mu_2)$. The proof is standard and is given as Lemma A.1 in Appendix A.1.

(5) T_C and U coincide on $\overline{\nabla\text{-span}}\{\mathbf{1}_{S_1}, f_1, f_2, \dots\}$. Observe that $T_C \mathbf{1}_{S_1} = \mathbf{1}_{S_2}$ and for any $B \in \mathcal{B}_{\mathbb{R}_+}$,

$$\{T_C f_j \in B\} = T_C(\{f_j \in B\}) = \{U f_j \in B\}, j = 1, 2, \dots,$$

where the first equality follows as in the case of linear mapping (e.g. see p452-454 [9]) and the second equality follows by (A.3). This shows that $T_C(f_j) = U(f_j), j = 1, 2, \dots$. Since T_C is a max-linear isometry by (4), T_C and U coincide on any finite positive max-linear combinations of $\mathbf{1}_{S_1}, f_1, f_2, \dots$ and hence on $\overline{\nabla\text{-span}}\{\mathbf{1}_{S_1}, f_1, f_2, \dots\}$.

(6) T_C is unique. Assume that there exists another max-linear isometry V from $L_+^\alpha(S_1, \sigma(C), \mu_1)$ to $L_+^\alpha(S_2, \mu_2)$ such that V and U agree on $\overline{\nabla\text{-span}}\{\mathbf{1}_{S_1}, f_1, f_2, \dots\}$. We will show that V and T_C coincide on $L_+^\alpha(S_1, \sigma(C), \mu_1)$. It is enough to show that, for any $A_0 \in \mathcal{B}_{\mathbb{R}_+}$ and any $A = \{(Uf_1, Uf_2, \dots) \in B_A\} \in \sigma(U(C))$, we have

$$\mu_2(\{T_C f \in A_0\} \cap A) = \mu_2(\{V f \in A_0\} \cap A).$$

Indeed, for any $f \in L_+^\alpha(S_1, \sigma(C), \mu_1)$ we have, by (5),

$$\begin{aligned} \mu_2(\{T_C f \in A_0\} \cap A) &= \mu_2(\{(T_C f, T_C f_1, T_C f_2, \dots) \in A_0 \times B_A\}) \\ &= \mu_1(\{(f, f_1, f_2, \dots) \in A_0 \times B_A\}) \end{aligned} \tag{A.5}$$

$$\begin{aligned} &= \mu_2(\{(V f, V f_1, V f_2, \dots) \in A_0 \times B_A\}) \\ &= \mu_2(\{(V f, U f_1, U f_2, \dots) \in A_0 \times B_A\}) \\ &= \mu_2(\{V f \in A_0\} \cap A) \end{aligned} \tag{A.6}$$

by using Lemma A.2 in (A.5) and (A.6). Hence, we have $\{T_C f \in A_0\} = \{V f \in A_0\}, \forall A_0 \in \mathcal{B}_{\mathbb{R}_+}$ and $\forall f \in L_+^\alpha(S_1, \sigma(C), \mu_1)$, which implies that T_C and V agree on $L_+^\alpha(S_1, \sigma(C), \mu_1)$.

To complete the proof, we define a max-linear isometry from $L_+^\alpha(S_1, \sigma(\mathcal{F}), \mu_1)$ onto $L_+^\alpha(S_2, \sigma(U(\mathcal{F})), \mu_2)$. Note that, for all $f \in L_+^\alpha(S_1, \sigma(\mathcal{F}), \mu_1)$ there exists a countable collection of functions $C = \{f_1, f_2, \dots\}$, such that $f \in \sigma(C)$. We therefore define $Tf = T_C f$. To check the consistency of this definition, suppose that $f \in \sigma(\tilde{C})$, for another countable collection of functions $\tilde{C} \subset \mathcal{F}$. Since $C \subset C \cup \tilde{C}$, by using (A.3) one can show that $T_C(A) = T_{C \cup \tilde{C}}(A)$ for every $A \in \sigma(C)$. Thus, $T_C f = T_{C \cup \tilde{C}} f$ and similarly $T_{\tilde{C}} f = T_{C \cup \tilde{C}} f$, which shows that T is well-defined. It is easy to see that T is induced by a measure preserving regular set isomorphism of $\sigma(\mathcal{F})$ onto $\sigma(U(\mathcal{F}))$. This is because that for every $A \in \sigma(\mathcal{F})$, we have $\mathbf{1}_A \in \sigma(C)$ with some countable collection $C \subset \mathcal{F}$. \square

Proof of Lemma 3.1. First, to show $\rho(F) = \rho(\overline{\nabla\text{-span}}(F))$, it suffices to show that $\rho(\overline{\nabla\text{-span}}(F)) \subset \rho(F)$. Observe that for any $f_i, g_i \in F, a_i \geq 0, b_i \geq 0, i \in \mathbb{N}$ and

$c > 0$, we have

$$\begin{aligned} \left\{ \frac{\bigvee_{i \in \mathbb{N}} a_i f_i}{\bigvee_{j \in \mathbb{N}} b_j g_j} \leq c \right\} &= \bigcap_{i \in \mathbb{N}} \left\{ \frac{a_i f_i}{\bigvee_{j \in \mathbb{N}} b_j g_j} \leq c \right\} \\ &= \bigcap_{i \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ \frac{a_i f_i}{\bigvee_{j \in \mathbb{N}} b_j g_j} < c + \frac{1}{n} \right\} = \bigcap_{i \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left\{ \frac{a_i f_i}{b_j g_j} < c + \frac{1}{n} \right\}. \end{aligned}$$

Hence $\rho(\overline{\nabla\text{-span}}(F)) \subset \rho(F)$. Next, $\rho(F) \subset \sigma(F)$ follows from the fact that

$$\{f_1/f_2 \leq c\} = \bigcup_{0 < q \in \mathbb{Q}} (\{f_1 \leq q\} \cap \{f_2 \geq q/c\}),$$

for any $f_1, f_2 \in F$ and $c > 0$, where \mathbb{Q} denotes the set of rational numbers. Finally, when $\mathbf{1}_S \in F$, $\rho(F) \supset \rho(f/\mathbf{1}_S : f \in F) = \sigma(F)$. As it is always true that $\rho(F) \subset \sigma(F)$, it follows $\rho(F) = \sigma(F)$. \square

Proof of Lemma 3.2. Suppose first that \mathcal{F} is separable, i.e., there exists a countable collection of functions $f_n \in \mathcal{F}, n \in \mathbb{N}$, such that $\mathcal{F} = \overline{\nabla\text{-span}}\{f_1, f_2, \dots\}$. Then, we have that

$$g = \bigvee_{n=1}^{\infty} \left(f_n / (2^n \|f_n\|_{L_+^\alpha(S, \mu)}) \right) \text{ belongs to } L_+^\alpha(S, \mu),$$

because

$$\|g\|_{L_+^\alpha(\mu)}^\alpha = \int \bigvee_{n=1}^{\infty} \frac{f_n^\alpha}{2^{n\alpha} \|f_n\|_{L_+^\alpha(S, \mu)}^\alpha} d\mu \leq \sum_{n=1}^{\infty} \int \frac{f_n^\alpha}{2^{n\alpha} \|f_n\|_{L_+^\alpha(S, \mu)}^\alpha} d\mu = \sum_n \frac{1}{2^{n\alpha}} < \infty.$$

Since \mathcal{F} is a max-linear space, we have $g \in \mathcal{F}$ and clearly g has full support in \mathcal{F} since for any $f \in \mathcal{F}$, $\text{supp}(f) \subset \bigcup_{n=1}^{\infty} \text{supp}(f_n) = \text{supp}(g) \pmod{\mu}$.

Next consider the case when μ is σ -finite. Let $\bar{\mu}$ be a finite measure equivalent to μ (i.e., $\bar{\mu} \ll \mu$ and $\mu \ll \bar{\mu}$). Now let $F \subset \mathcal{F}$ be any arbitrary countable collection of functions in \mathcal{F} , set $s(F) := \bar{\mu} \left(\bigcup_{f \in F} \text{supp}(f) \right)$ and define $s := \sup_{F \in \mathcal{F}} s(F)$. Thus, consider a sequence $F_n \subset \mathcal{F}, n \in \mathbb{N}$ of countable collections of functions, such that $s(F_n) \uparrow s$ as $n \rightarrow \infty$. Let $C = \bigcup_{n \in \mathbb{N}} F_n$ and observe that C is countable. Then by the first part of the proof, there exists $g \in \overline{\nabla\text{-span}}(C)$ with full support in $\overline{\nabla\text{-span}}(C)$, since $\bar{\mu} \left(\bigcup_{f \in C} \text{supp}(f) \setminus \text{supp}(g) \right) = 0$ implies $\mu \left(\bigcup_{f \in C} \text{supp}(f) \setminus \text{supp}(g) \right) = 0$. The function g has also full support in \mathcal{F} . Indeed, if there exists a function $f_0 \in \mathcal{F}$ such that $\bar{\mu}(\text{supp}(f_0) \setminus \text{supp}(g)) = \epsilon > 0$, then $f_0 \notin C$ and $\lim_{n \rightarrow \infty} s(F_n \cup \{f_0\}) \geq s + \epsilon > s$, which is a contradiction. This completes the proof of the lemma. \square

Proof of Lemma 3.3. Let $g_0 = Uf_0$ and let $g_1 = Uf_1$ for an arbitrary $f_1 \in \mathcal{F}$. We clearly have that $f_2 := f_0 \vee f_1$ and $g_2 := g_0 \vee g_1 = Uf_2$ have full supports in $\overline{\nabla\text{-span}}\{f_1, f_2\}$ and $\overline{\nabla\text{-span}}\{g_1, g_2\}$, respectively. To prove the result, it is enough to show that $\mu_2(\text{supp}(g_1) \setminus \text{supp}(g_0)) = 0$, or equivalently, $\mu_2(\text{supp}(g_2) \setminus \text{supp}(g_0)) = 0$.

Consider the finite measures

$$\nu_1 = f_2^\alpha d\mu_1 \quad \text{and} \quad \nu_2 = g_2^\alpha d\mu_2, \tag{A.7}$$

restricted to the spaces $(\text{supp}(f_2), \mathcal{B}_{S_1}|_{\text{supp}(f_2)})$ and $(\text{supp}(g_2), \mathcal{B}_{S_2}|_{\text{supp}(g_2)})$. Now, define

$$V(a\mathbf{1}_{\text{supp}(f_2)} \vee b(f_0/f_2)) := a\mathbf{1}_{\text{supp}(g_2)} \vee b(g_0/g_2) \quad \forall a, b \geq 0.$$

Observe that

$$\begin{aligned} \int_{\text{supp}(f_2)} \left(a \mathbf{1} \vee \lambda b f_0 \frac{1}{f_2} \right)^\alpha d\nu_1 &= \int_{S_1} \left(a f_2 \vee \lambda b f_0 \right)^\alpha d\mu_1 \\ &= \int_{S_2} \left(a g_2 \vee \lambda b g_0 \right)^\alpha d\mu_2 = \int_{\text{supp}(g_2)} \left(a \mathbf{1} \vee \lambda b g_0 \frac{1}{g_2} \right)^\alpha d\nu_2. \end{aligned}$$

This shows that $V : \overline{\text{span}}\{\mathbf{1}_{\text{supp}(f_2)}, f_0/f_2\} \rightarrow L_+^\alpha(\text{supp}(g_2), \nu_2)$ is a max-linear isometry mapping $\mathbf{1}_{\text{supp}(f_2)}$ to $\mathbf{1}_{\text{supp}(g_2)}$. Thus, by Lemma A.2, we obtain

$$\nu_1(\{f_0/f_2 = 0\}) = \nu_2(\{V(f_0/f_2) = 0\}) = \nu_2(\{g_0/g_2 = 0\}) = 0,$$

Since $\nu_1(\{f_0/f_2 = 0\}) = \nu_1(\text{supp}(f_2) \setminus \text{supp}(f_0)) = 0$, we also have that $\nu_2(g_0/g_2 = 0) = \nu_2(\text{supp}(g_2) \setminus \text{supp}(g_0)) = 0$. This, in view of (A.7), implies that $\mu_2(\text{supp}(g_2) \setminus \text{supp}(g_0)) = 0$ and hence $\mu_2(\text{supp}(g_1) \setminus \text{supp}(g_0)) = 0$, since $\text{supp}(g_1) \subset \text{supp}(g_2)$. We have thus shown that for an arbitrary $f_1 \in \mathcal{F}$, $\mu_2(\text{supp}(Uf_1) \setminus \text{supp}(g_0)) = 0$, which shows that $Uf_0 = g_0$ has full support in $U(\mathcal{F})$ (see Definition 3.4). \square

Proof of Theorem 3.2. Let $f_0 \in \mathcal{F}$ be a function with full support in \mathcal{F} , i.e., $\text{supp}(f_0) = S_1$ (Lemma 3.2). Define $\mathcal{F}_0 := \{f \cdot (1/f_0), f \in \mathcal{F}\}$. Since $f_0/f_0 = \mathbf{1}_{S_1} \in \mathcal{F}_0$, it follows that

$$\sigma(\mathcal{F}_0) = \rho(\mathcal{F}_0) = \rho(\mathcal{F}), \quad (\text{A.8})$$

where the second equality follows from the fact that $f_1/f_2 = (f_1/f_0)/(f_2/f_0) \bmod \mu_1$, for all $f_1, f_2 \in \mathcal{F}$, since f_0 has full support in \mathcal{F} . Therefore, any element $rf \in \mathcal{R}_{e,+}(\mathcal{F})$, $r \in \mathcal{R}_+(\mathcal{F})$ and $f \in \mathcal{F}$, can be represented as follows:

$$rf = (rf \cdot (1/f_0)) f_0 = r_0 f_0,$$

where $r_0 = rf \cdot (1/f_0) = r \cdot (f/f_0)$ is a $\rho(\mathcal{F})$ -measurable and hence $\sigma(\mathcal{F}_0)$ -measurable function. Hence, we have that

$$\mathcal{R}_{e,+}(\mathcal{F}) = \{r_0 f_0 \in L_+^\alpha(S_1, \mu_1), r_0 \geq 0, r_0 \in \sigma(\mathcal{F}_0)\}.$$

Next, introduce the measures $d\mu_{1,f_0} = f_0^\alpha d\mu_1$ and $d\mu_{2,f_0} = (Uf_0)^\alpha d\mu_2$, and observe that both of them are finite. We thus have that \mathcal{F}_0 is a max-linear sub-space of $L_+^\alpha(S_1, \mu_{1,f_0})$ and similarly $\mathcal{G}_0 := \{Uf \cdot (1/Uf_0), f \in \mathcal{F}\}$ is a max-linear sub-space $\subset L_+^\alpha(S_2, \mu_{2,f_0})$. It is easy to check that

$$U_0 : \mathcal{F}_0 \rightarrow \mathcal{G}_0, \quad \text{defined by } U_0(f) := U(f \cdot f_0) \cdot (1/Uf_0), \quad f \in \mathcal{F}_0$$

is a max-linear isometry from $\mathcal{F}_0 \subset L_+^\alpha(S_1, \mu_{1,f_0})$ to $\mathcal{G}_0 \subset L_+^\alpha(S_2, \mu_{2,f_0})$. Note, however, that these two L_+^α -spaces involve finite measures and $U_0 \mathbf{1}_{S_1} = \mathbf{1}_{\text{supp}(Uf_0)}$. Thus, by Theorem 3.1, we obtain that U_0 has a unique extension to a max-linear isometry

$$T : L_+^\alpha(S_1, \sigma(\mathcal{F}_0), \mu_{1,f_0}) \rightarrow L_+^\alpha(\text{supp}(Uf_0), \sigma(\mathcal{G}_0)|_{\text{supp}(Uf_0)}, \mu_{2,f_0}),$$

which is induced by a measure preserving regular set isomorphism T from $\sigma(\mathcal{F}_0)$ onto $\sigma(\mathcal{G}_0)$.

We can now construct the desired extension \overline{U} of the max-linear isometry U . Consider the mappings

$$M : \mathcal{R}_{e,+}(\mathcal{F}) \rightarrow L_+^\alpha(S_1, \sigma(\mathcal{F}_0), \mu_{1,f_0})$$

defined by $Mf := f \cdot (1/f_0)$, $\forall f \in \mathcal{R}_{e,+}(\mathcal{F})$ and

$$N : L_+^\alpha(\text{supp}(Uf_0), \sigma(\mathcal{G}_0)|_{\text{supp}(Uf_0)}, \mu_{2,f_0}) \rightarrow L_+^\alpha(S_2, \mu_2)$$

defined by $Ng := g \cdot (Uf_0)$, $\forall g \in L_+^\alpha(\text{supp}(Uf_0), \sigma(\mathcal{G}_0)|_{\text{supp}(Uf_0)}, \mu_{2,f_0})$. Note that both mappings M and N are *one-to-one* and that M is trivially *onto*. We will now show that N is also *onto*. Indeed, as in (A.8), we have that

$$\sigma(\mathcal{G}_0) = \rho(\mathcal{G}_0) = \rho(U(\mathcal{F})). \quad (\text{A.9})$$

Consider an arbitrary $g \in \mathcal{R}_{e,+}(U(\mathcal{F}))$, and note that $g = rU(f)$, with some $r \in \rho(U(\mathcal{F}))$ and $f \in \mathcal{F}$. We have that $g = \tilde{r}U(f_0)$ with $\tilde{r} = rU(f)/U(f_0)$, since Uf_0 has full support in $U(\mathcal{F})$ (Lemma 3.3). By (A.9), we have that \tilde{r} is $\rho(U(\mathcal{F}))$ and hence $\sigma(\mathcal{G}_0)$ -measurable, and since $g = rU(f) \in L_+^\alpha(S_2, \mu_2)$, it follows that $\tilde{r} \in L_+^\alpha(S_2, \sigma(\mathcal{G}_0), \mu_{2,f_0})$. This shows that $N(\tilde{r}) = rU(f) = g$, and since $g \in \mathcal{R}_{e,+}(U(\mathcal{F}))$ was arbitrary, it follows that N is *onto* $\mathcal{R}_{e,+}(U(\mathcal{F}))$.

At last, we define

$$\overline{U} := NTM : \mathcal{R}_{e,+}(\mathcal{F}) \rightarrow L_+^\alpha(S_2, \mu_2).$$

We will complete the proof by verifying that \overline{U} satisfies (3.1) and (3.2) as well as the fact that \overline{U} is *onto* and *unique*. To prove (3.1), observe that

$$\overline{U}(rf) = NTM(rf \cdot (1/f_0) \cdot f_0) = NT(rf \cdot (1/f_0)) = (Uf_0)T(r)T(f \cdot (1/f_0)),$$

where the last equality follows from the fact that $T(f_1f_2) = T(f_1)T(f_2)$, for any two measurable functions f_1 and f_2 (Lemma A.1). Since $T(f \cdot (1/f_0)) = U_0(f \cdot (1/f_0)) = U(f)/U(f_0)$, we obtain that

$$\overline{U}(rf) = (Uf_0)T(r)U(f)/U(f_0) = T(r)U(f),$$

which yields (3.1).

To prove (3.2), note that for all $A \in \rho(U(\mathcal{F}))$, we have

$$\begin{aligned} (\mu_{1,f} \circ T^{-1}) A &= \int_{S_1} (\mathbf{1}_{T^{-1}A} f)^\alpha d\mu_1 \\ &= \int_{S_2} T(\mathbf{1}_{T^{-1}A})^\alpha T(f)^\alpha d\mu_2 = \int_{S_2} \mathbf{1}_A U(f)^\alpha d\mu_2, \end{aligned}$$

which is equivalent to Relation (3.2).

Now, the extension $\overline{U} = NTM$ is *onto* $\mathcal{R}_{e,+}(U(\mathcal{F}))$ because so are the mappings M, N and T . Finally, to prove the uniqueness of \overline{U} , suppose that there exists another max-linear isometry, $V : \mathcal{R}_{e,+}(\mathcal{F}) \rightarrow L_+^\alpha(S_2, \mu_2)$, extending U . By the definitions of M and N , we have that $N^{-1}VM^{-1}$ is a max-linear isometry from $L_+^\alpha(S_1, \sigma(\mathcal{F}_0), \mu_{1,f_0})$ to $L_+^\alpha(\text{supp}(Uf_0), \sigma(\mathcal{G}_0)|_{\text{supp}(Uf_0)}, \mu_{2,f_0})$. We also have that

$$N^{-1}VM^{-1}(f/f_0) = N^{-1}Vf = N^{-1}(Uf) = Uf/(Uf_0), \quad \text{for all } f \in \mathcal{F},$$

which shows that $N^{-1}VM^{-1}$ coincides with U_0 on \mathcal{F}_0 . Since U_0 has a unique extension $T : L_+^\alpha(S_1, \sigma(\mathcal{F}_0), \mu_{1,f_0}) \rightarrow L_+^\alpha(\text{supp}(Uf_0), \sigma(\mathcal{G}_0)|_{\text{supp}(Uf_0)}, \mu_{2,f_0})$, we obtain that $N^{-1}VM^{-1} = T$, which implies $V = NTM \equiv \overline{U}$. This completes the proof of the theorem. \square

Proof of Lemma 3.4. Fix $f_0 \in F$ with full support. Since f_0 is $\sigma(F)$ -measurable, so is $1/f_0$. Now for any $f \in L_+^\alpha(S, \mu)$, f is $\sigma(F)$ -measurable. Observe that $\rho(F) \subset \sigma(F) \subset \mathcal{B}_S$, whence, by $\rho(F) \sim \mathcal{B}_S \mod \mu$, $\rho(F) \sim \sigma(F) \sim \mathcal{B}_S \mod \mu$. Hence, $f \cdot (1/f_0)$ is \mathcal{B}_S -measurable. Thus $f = (f \cdot (1/f_0))f_0 \in L_+^\alpha(S, \mu)$. \square

A.2 Proofs for Sections 5 and 6

Proof of Proposition 5.2. To prove part (i), observe that since μ is σ -finite, it is enough to focus on the case when μ is a probability measure: $\mu(S) = 1$. Thus, $\{f_t(s)\}_{t \in T}$ may be viewed as a stochastic process, defined on the probability space (S, \mathcal{B}_S, μ) .

Note that $L_+^\alpha(S, \mu)$ equipped with the metric $\rho_{\mu, \alpha}(f, g) = \int_S |f^\alpha - g^\alpha| d\mu$, is a complete separable metric space. Furthermore, $\rho_{\mu, \alpha}$ metrizes the convergence in probability in the space (S, μ) . Therefore, Theorem 3 of Cohn [4] (see also Proposition 9.4.4 in [32]) implies that the stochastic process $f = \{f_t(s)\}_{t \in T}$ has a measurable modification *if and only if* the map $h_f : t \mapsto [f_t]$ is Borel-measurable and has separable range $h_f(T)$. Here $[f]$ denotes the class of all $L_+^\alpha(\mu)$ -functions, equal to f , μ -a.e..

Similarly, $X = \{X_t\}_{t \in T}$ has a measurable modification *if and only if* $h_X : t \mapsto [X_t]$ is Borel-measurable and has separable range $h_X(T)$, where $[X_t] \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a metric, which metrizes the convergence in probability. Here $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ denotes the collection of equivalence classes of random variables, with respect to the relation of almost sure equality. We focus on the set $\mathcal{M} = \{[\xi] : \xi = \int_S g dM_\alpha, g \in L_+^\alpha(S, \mu)\}$, which is a closed subset of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the convergence in probability. Theorem 2.1 of [35], shows that since $(L_+^\alpha(S, \mu), \rho)$ is complete and separable, so is \mathcal{M} with respect to the metric:

$$\rho_{\mathcal{M}}(\xi, \eta) := 2\|\xi \vee \eta\|_\alpha^\alpha - \|\xi\|_\alpha^\alpha - \|\eta\|_\alpha^\alpha.$$

Furthermore, $\rho_{\mathcal{M}}$ metrizes the convergence in probability and we have

$$\rho_{\mathcal{M}}(\xi, \eta) = \int_S |f^\alpha - g^\alpha| d\mu \equiv \rho(f, g), \quad (\text{A.10})$$

for all $\xi = \int_S f dM_\alpha$ and $\eta = \int_S g dM_\alpha$, with $f, g \in L_+^\alpha(S, \mu)$.

Now, the separability of $L_+^\alpha(S, \mu)$ and \mathcal{M} implies the separability of the ranges $h_f(T) \subset L_+^\alpha(S, \mu)$ and $h_X(T) \subset \mathcal{M}$, respectively. On the other hand, the equivalence (A.10) of the two metrics $\rho_{\mathcal{M}}$ and ρ implies that $h_f : T \rightarrow L_+^\alpha(S, \mu)$ is Borel-measurable *if and only if* $h_X : T \rightarrow \mathcal{M}$ is Borel-measurable. This, in view of Theorem 3 of Cohn [4], yields (i).

In view of Proposition 2.1, to establish (ii), we should show that any measurable α -Fréchet process X satisfies Condition S. As argued above, the map $h_X : t \mapsto [X_t]$ has a separable range in the metric space $\mathcal{L}_0^0(\Omega, \mathcal{F}, \mathbb{P})$. Hence, there exists a countable set $T_0 \subset T$, such that for all $t \in T$, for some $t_n \in T_0$, we have $X_{t_n} \xrightarrow{P} X_t$, as $n \rightarrow \infty$. This shows that the process X is separable in probability (satisfies Condition S, see Definition 2.2) and the proof is complete. \square

Proof of Theorem 5.2. Part (ii) follows immediately from (5.6). To prove (i), consider another measurable representation $\{f_t^{(2)}\}_{t \in T} \subset L_+^\alpha(S_2, \mu_2)$ of the same process $\{X_t\}_{t \in T}$. We show that $\{f_t^{(2)}\}_{t \in T}$ also admits a co-spectral decomposition and, letting the corresponding decomposition of the process be

$$\{X_t\}_{t \in T} \stackrel{d}{=} \left\{ \widehat{X}_t^{(1)} \vee \dots \vee \widehat{X}_t^{(n)} \right\}_{t \in T}, \quad (\text{A.11})$$

we have

$$\{X_t^{(j)}\}_{t \in T} \stackrel{d}{=} \{\widehat{X}_t^{(j)}\}_{t \in T}, 1 \leq j \leq n. \quad (\text{A.12})$$

Let $\{f_t^{(1)}\}_{t \in T} \subset L_+^\alpha(S_1, \mu_1)$ denote the representation in assumption, which admits a co-spectral decomposition w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$. Without specification, the following arguments hold for both $i = 1, 2$.

First, by Proposition 5.2, the process X has the representation in (2.4), and hence it has a minimal representation with standardized support $\{f_t(s)\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$ by Theorem 4.1. This representation can be also chosen to be jointly measurable. By (4.6) in Theorem 4.2, we have

$$f_t^{(i)}(s) = h_i(s)f_t(\Phi_i(s)) =: \widetilde{f}_t^{(i)}(s), \mu_i\text{-a.e.}, \forall t \in T, \quad (\text{A.13})$$

where $h_i : S_i \rightarrow \mathbb{R}_+ \setminus \{0\}$ and Φ_i from S_i onto $S_{I,N}$ are both measurable. Since $(t, s) \mapsto f_t(s)$ is measurable, it follows that $\widetilde{f}_t^{(i)}(s)$ is jointly measurable modification of $f_t^{(i)}(s)$. Consider the sets

$$N^{(i)} := \left\{ (t, s) : f_t^{(i)}(s) \neq \widetilde{f}_t^{(i)}(s) \right\}.$$

By (A.13), we have that $\mu_i(N_t^{(i)}) = 0, \forall t \in T$, where $N_t^{(i)} = \{s : (t, s) \in N^{(i)}\}$. Thus, by Fubini's Theorem, there exists $\widetilde{S}_i \subset S_i$ such that $\mu_i(S_i \setminus \widetilde{S}_i) = 0$ and for all $s \in \widetilde{S}_i$, $\widetilde{f}_t^{(i)}(s) = f_t^{(i)}(s), \lambda\text{-a.e.}$

The argument above implies that

$$f_t^{(i)}(s) = h_i(s)f_t \circ \Phi_i(s), \forall (t, s) \in T \times \widetilde{S}_i. \quad (\text{A.14})$$

Now, suppose S_1 has a co-spectral decomposition $S_1 = \bigcup_{j=1}^n S_1^{(j)} \bmod \mu_1$. We show that this induces a co-spectral decomposition of $S_{I,N}$. Without loss of generality, assume that $S_1^{(j)} \subset \widetilde{S}_1, 1 \leq j \leq n$. Set

$$S_{I,N}^{(j)} := \Phi_1(S_1^{(j)}), 1 \leq j \leq n \text{ and } S_{I,N}^{(0)} := S_{I,N} \setminus \bigcup_{j=1}^n S_{I,N}^{(j)}. \quad (\text{A.15})$$

By (A.14), $S_{I,N}^{(j)} \subset \{s : f_t(s) \in \mathcal{P}_j\}, 1 \leq j \leq n$. Note that the assumption $S_1^{(j)} \cap S_1^{(k)} \subset \{s \in S_1 : f_t^{(1)}(s) \equiv 0\}$ implies that $S_{I,N}^{(j)} \cap S_{I,N}^{(k)} \subset \{s \in S_{I,N} : f_t(s) \equiv 0\}$, for all $1 \leq j < k \leq n$. Moreover, $\Phi_1^{-1}(S_{I,N}^{(0)}) \subset S_1 \setminus \bigcup_{j=1}^n S_1^{(j)}$, whence $\lambda_{I,N}(S_{I,N}^{(0)}) = 0$. We have thus shown that $\{S_{I,N}^{(j)}\}_{1 \leq j \leq n}$ is a co-spectral decomposition of $\{f_t\}_{t \in T} \subset L_+^\alpha(S_{I,N}, \lambda_{I,N})$, w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$.

Next, we show that for any spectral representation $\{f_t^{(2)}\}_{t \in T} \subset L_+^\alpha(S_2, \mu_2)$, there exists a co-spectral decomposition of S_2 w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$. Indeed, the decomposition is induced by setting $S_2^{(j)} := \Phi_2^{-1}(S_{I,N}^{(j)}) \cap \widetilde{S}_2, 1 \leq j \leq n$. One can easily verify that $\{S_2^{(j)}\}_{1 \leq j \leq n}$ is a co-spectral decomposition w.r.t. $\{\mathcal{P}_j\}_{1 \leq j \leq n}$.

Finally, by the construction of $\{S_i^{(j)}\}_{1 \leq j \leq n}, i = 1, 2$ above, we have

$$\lambda_{I,N} \left(\Phi_i(S_i^{(j)}) \triangle S_{I,N}^{(j)} \right) = 0, \quad \forall 1 \leq j \leq n. \quad (\text{A.16})$$

Note that (A.14) induces a max-linear isometry from $L_+^\alpha(S_{I,N}, \lambda_{I,N})$ to $L_+^\alpha(S_i, \mu_i)$. Combining with (A.16) and Remark 4.3, we have

$$\left\{ \int_{S_i^{(j)}}^e f_t^{(i)} dM_\alpha^{(i)} \right\}_{t \in T} \stackrel{d}{=} \left\{ \int_{S_{I,N}^{(j)}}^e f_t dM_\alpha \right\}_{t \in T}, 1 \leq j \leq n.$$

This implies (A.12). \square

Proof of Theorem 6.1. This result can be established by following closely the proof of Theorem 3.1 in [27] and replacing the linear combination $g_n = \sum_{i=1}^n c_{ni} f_{ni}$ therein by the max-linear combination $g_n = \bigvee_{i=1}^n c_{ni} f_{ni} \in \vee\text{-span}\{f_t : t \in T\}$. For the completeness, we provide the details next.

Suppose $\{f_t\}_{t \in T}$ is minimal. Then, for any $\tau \in T$, by stationarity $\{f_{t+\tau}\}_{t \in T}$ is also a minimal representation of the same α -Fréchet process. By applying Corollary 4.1, there exist a one-to-one and onto measurable function $\Phi_\tau : S_{I,N} \rightarrow S_{I,N}$ and a measurable function $h_\tau : S_{I,N} \rightarrow \mathbb{R}_+ \setminus \{0\}$ such that for each $t \in T$,

$$f_{t+\tau}(s) = h_\tau(s) (f_t \circ \Phi_\tau)(s), \quad \lambda_{I,N}\text{-a.e.} \quad (\text{A.17})$$

and

$$\frac{d(\lambda_{I,N} \circ \Phi_\tau)}{d\lambda_{I,N}}(s) = h_\tau(s)^\alpha, \quad \lambda_{I,N}\text{-a.e.} \quad (\text{A.18})$$

Since, for every $t, \tau_1, \tau_2 \in T$, we have two ways expressing $f_{t+\tau_1+\tau_2}$:

$$f_{t+\tau_1+\tau_2} = f_{(t+\tau_1)+\tau_2} = (h_{\tau_2})(f_{t+\tau_1} \circ \Phi_{\tau_2}) = (h_{\tau_2})(h_{\tau_1} \circ \Phi_{\tau_2})(f_t \circ \Phi_{\tau_1} \circ \Phi_{\tau_2}), \lambda_{I,N}\text{-a.e.}$$

and

$$f_{t+\tau_1+\tau_2} = (h_{\tau_1+\tau_2})(f_t \circ \Phi_{\tau_1+\tau_2}), \lambda_{I,N}\text{-a.e.},$$

it follows, by the uniqueness of Φ_τ and h_τ , that for every $\tau_1, \tau_2 \in T$,

$$h_{\tau_1+\tau_2} = (h_{\tau_2})(h_{\tau_1} \circ \Phi_{\tau_2}), \lambda_{I,N}\text{-a.e.}, \quad (\text{A.19})$$

and

$$\Phi_{\tau_1+\tau_2} = \Phi_{\tau_1} \circ \Phi_{\tau_2}, \lambda_{I,N}\text{-a.e.} \quad (\text{A.20})$$

To complete the proof, we will establish a modification ϕ of Φ such that ϕ is measurable on $T \times S_{I,N}$ and

$$\Phi_t(s) = \phi(t, s), \lambda_{I,N}\text{-a.e.}, \forall t \in T.$$

If $T = \mathbb{Z}$, then one can modify $\{\Phi_t\}_{t \in T}$ to have (A.20) hold everywhere for all τ_1, τ_2 , making $\{\Phi_t\}_{t \in T}$ a flow. When $T = \mathbb{R}$, by Theorem 1 in [20], in order for $\{\Phi_t\}_{t \in T}$ to have a measurable version $\{\phi_t\}_{t \in T}$, it is enough to check that the map

$$t \mapsto \tilde{\nu}([\Phi_t^{-1}(B)])$$

is measurable for every finite measure $\tilde{\nu}$ on $\mathcal{B}_{\lambda_{I,N}}$ (the measure algebra induced by $(\mathcal{B}_{S_{I,N}}, \lambda_{I,N})$). It is clear that $\tilde{\nu}$ defines a finite measure ν on $\mathcal{B}_{S_{I,N}}$ such that $\nu(B) = \tilde{\nu}([B])$ and we have $\nu \ll \lambda_{I,N}$. Put $k = d\nu/d\lambda_{I,N}$. It is equivalent to show that

$$t \mapsto \int_{S_{I,N}} \mathbf{1}_B(\Phi_t(s)) k(s) \lambda_{I,N}(ds) \quad (\text{A.21})$$

is measurable for each $B \in \mathcal{B}_{S_{I,N}}$. Indeed, it is enough to show that $(t, s) \mapsto \mathbf{1}_B(\Phi_t(s))$ is a measurable function of (t, s) for each $B \in \mathcal{B}_{S_{I,N}}$. Choose a function $g = f_{I,N}$ defined in (4.5) and $g_n = \bigvee_{i=1}^n c_{ni} f_{t_{ni}} \in \vee\text{-span}\{f_t, t \in T\}$, such that $g_n \rightarrow g$, $\lambda_{I,N}$ -a.e.. In view of (6.1), for each $\tau \in T$,

$$h_\tau(s)g_n \circ \Phi_\tau(s) = \bigvee_{i=1}^n c_{ni} f_{t_{ni}+\tau}(s), \lambda_{I,N}\text{-a.e. } s \in S_{I,N}.$$

Observe that the r.h.s. is a measurable function of (τ, s) for each $n \in \mathbb{N}$ and the l.h.s. converges $\lambda_{I,N}$ -a.e. as $n \rightarrow \infty$, for all $t \in T$. It follows that there exists a measurable function $(\tau, s) \mapsto g_\tau(s)$ such that, for each $\tau \in T$,

$$h_\tau(s)g \circ \Phi_\tau(s) = g_\tau(s), \lambda_{I,N}\text{-a.e.} \quad (\text{A.22})$$

Now, observe that since $\{f_t\}_{t \in T}$ is minimal, for every $B \in \mathcal{B}_{S_{I,N}}$ there exist $t_1, t_2, \dots \in T$ and $A \in \mathbb{R}^\mathbb{N}$ such that $B = \{s : (f_{t_1}(s)/g(s), f_{t_2}(s)/g(s), \dots) \in A\} \pmod{\lambda_{I,N}}$. Note that (A.17) and (A.22) imply

$$\frac{f_t \circ \Phi_\tau(s)}{g \circ \Phi_\tau(s)} = \frac{f_{t+\tau}(s)}{g_\tau(s)}, \lambda_{I,N}\text{-a.e.}$$

It follows that

$$\mathbf{1}_B(\Phi_\tau(s)) = \mathbf{1}_A(f_{t_1+\tau}(s)/g_\tau(s), f_{t_2+\tau}(s)/g_\tau(s), \dots), \lambda_{I,N}\text{-a.e. } s \in S_{I,N}.$$

We have thus shown that the map in (A.21) is measurable. \square

Proof of Proposition 6.1. (i) The fact that $\{f_t^{(2)}\}_{t \in T}$ is another spectral representation of $\{X_t\}_{t \in T}$ can be verified by checking

$$\left\| \bigvee c_j f_{t_j}^{(2)} \right\|_{L_+^\alpha(S_2, \mu_2)} = \left\| \bigvee c_j f_{t_j}^{(1)} \right\|_{L_+^\alpha(S_1, \mu_1)}.$$

(ii) By Corollary 4.1, there exists measurable and invertible point mapping $\Phi : S_2 \rightarrow S_1$ such that we have two different ways relating $f_{t+\tau}^{(2)}$ and $f_t^{(1)}$:

$$f_{t+\tau}^{(2)} = \left(\frac{d(\mu_2 \circ \phi_\tau^{(2)})}{d\mu_2} \right)^\alpha f_t^{(2)} \circ \phi_\tau^{(2)} = \left(\frac{d(\mu_1 \circ \Phi \circ \phi_\tau^{(2)})}{d\mu_2} \right)^\alpha f_t^{(1)} \circ \Phi \circ \phi_\tau^{(2)}, \mu_2\text{-a.e.},$$

and

$$f_{t+\tau}^{(2)} = \left(\frac{d(\mu_1 \circ \Phi)}{d\mu_2} \right)^\alpha f_{t+\tau}^{(2)} \circ \phi_\tau^{(2)} = \left(\frac{d(\mu_1 \circ \phi_\tau^{(1)} \circ \Phi)}{d\mu_2} \right)^\alpha f_t^{(1)} \circ \phi_\tau^{(1)} \circ \Phi, \mu_2\text{-a.e.}$$

By the uniqueness of determining flow, we have $\phi_\tau^{(1)} \circ \Phi = \Phi \circ \phi_\tau^{(2)}$, μ_2 -a.e.. \square

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